Problems #4

Problem 1. (1 XP) Prove that $2(-4)^n\binom{1/2}{n+1} = \frac{1}{n+1}\binom{2n}{n}$.

Solution. We have

$$\begin{aligned} 2(-4)^n\binom{1/2}{n+1} &= 2(-4)^n\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{(n+1)!} \\ &= 2^n\frac{1\cdot 3\cdots(2n-1)}{(n+1)!} \\ &= 2^n\frac{(2n)!}{2\cdot 4\cdots(2n)(n+1)!} \\ &= \frac{(2n)!}{n!(n+1)!} \\ &= \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

□

Problem 2. (2 XP) Let C_n be the nth Catalan number.

(a) Show that C_n counts the number of "legal" expressions that can be formed using n pairs of parentheses. For instance, $C_3 = 5$ because we have the possibilities ((())), (()()), (())(), ()(()), ()()().

(b) **(bonus; 2 XP extra)** Show that C_n also counts the number of permutations of $\{1, 2, \ldots, n\}$ that are 123-avoiding. That is, those permutations $\pi_1\pi_2\ldots\pi_n$ such that we do not have $i < j < k$ with $\pi_i < \pi_j < \pi_k$.

For instance, the 123-avoiding permutations of $\{1, 2, 3, 4\}$ are the $C_4 = 14$ permutations 1432, 2143, 2413, 2431, 3142, 3214, 3241, 3412, 3421, 4132, 4213, 4231, 4312, 4321. On the other hand, 2314 is not 123-avoiding because it contains 234 as a substring.

Solution.

(a) Note that each expression is of the form $(A)B$, where A and B are themselves legal parenthetical expressions. The expression A can involve anywhere from $k = 0$ to $k = n-1$ pairs of parentheses, in which case B is formed from the remaining $n-1-k$. In terms of a formula, this means that, for $n \geqslant 1$,

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k},$$

which is Segner's recurrence relation. From it, as well as $C_0 = 1$, we derived the generating function for the Catalan numbers in class.

(b) In fact, we can replace 123 with any permutation $\sigma \in S_3$. That is, the number of σ-avoiding permutations is C_n.

See, for instance, Stanley's recent book on Catalan numbers for this and many more interpretations of the Catalan numbers.

Here, we only note that the study and enumeration of pattern-avoiding permutations is a very active and surprisingly large enterprise within combinatorics, with many open problems sparking people's interests. □

Exploring Sage

Problem 3. (2 XP extra) Explore `CFiniteSequences` in Sage.

It turns out that C-finite sequences are closed under the Hadamard product, that is, if a_n and b_n are Cfinite, then the product $c_n = a_n b_n$ is C-finite. Unfortunately, this closure property is not yet implemented in Sage. Nevertheless, find a (possibly heuristic) way to find the generating function of F_n^2, the square of the Fibonacci numbers.

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Solution. One possibility is to use the guess functionality:

```
Sage] C.<x> = CFiniteSequences(QQ)
Sage] fibo_sq = C.guess([fibonacci(n)^2 for n in [0..10]])
Sage] fibo_sq.ogf()
```

$$\frac{-x^2+x}{x^3-2\,x^2-2\,x+1}$$

```
Sage] [fibo_sq[n] for n in [0..10]]
```

$$[0, 1, 1, 4, 9, 25, 64, 169, 441, 1156, 3025]$$

As we will see later in class, we know *a priori* that the squares of the Fibonacci numbers satisfy a recursion with constant coefficients of order at most three. Our guessing therefore is guaranteed to produce the correct generating function. □

Problem 4. (2 XP extra) Jeff Lagarias proved in 2002 that the Riemann hypothesis is equivalent to

$$\sigma(n) < H_n + \ln(H_n)e^{H_n}$$

for all $n > 1$. Here, $\sigma(n) = \sum_{d|n} d$ is the sum of the divisors of n. Obtain numerical evidence using Sage by verifying that the inequality holds for small n. Also, make plots to get a visual impression.

Solution. Here is just some ideas how to get started investigating this inequality.

```
Sage] [sigma(n) for n in [1..11]]
```

$$[1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 12]$$

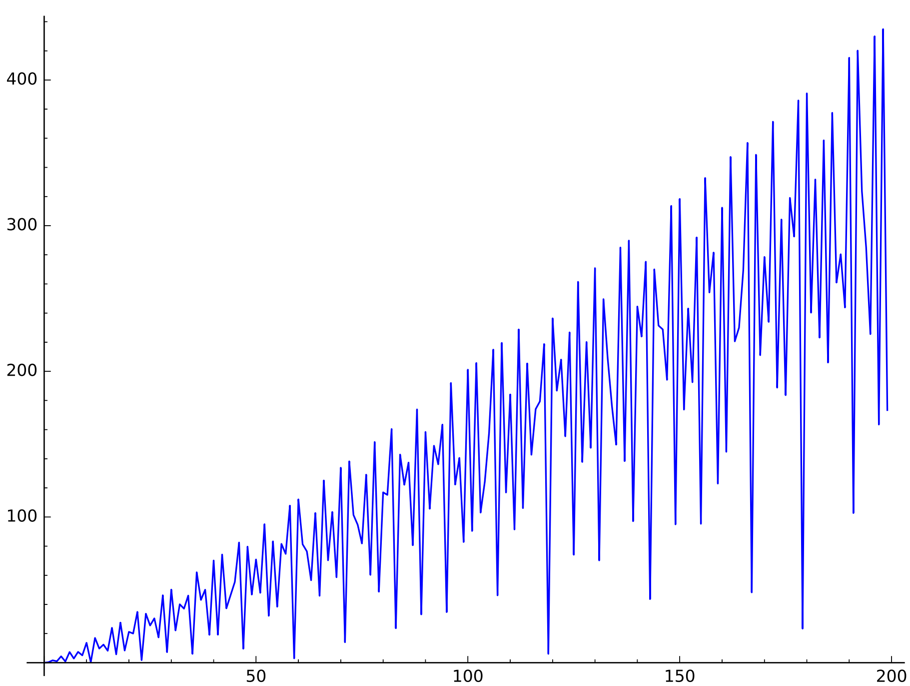
```
Sage] def H(n):
        return sum(1/k for k in [1..n])
Sage] [H(n) for n in [1..11]]
```

$$\left[1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{49}{20}, \frac{363}{140}, \frac{761}{280}, \frac{7129}{2520}, \frac{7381}{2520}, \frac{83711}{27720}\right]$$

```
Sage] [(H(n)+ln(H(n))*exp(H(n))-sigma(n)).n(20) for n in [1..11]]
```

$$[0.00000, 0.31717, 1.6245, 0.97797, 4.3823, 0.83418, 7.3293, 2.8633, 7.4326, 5.0338, 13.664]$$

```
Sage] list_plot([(H(n)+ln(H(n))*exp(H(n))-sigma(n)).n() for n in [1..200]], plotjoined=True)
```



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To collect 10^6 USD, it only remains to show that none of these spikes dip below the x-axis. $\square$

Problem 5. (1 XP extra) Define the following function $A(n)$ in Sage:

```
Sage] def A(n):
          if n==0: return 1
          return 2*(2*n-1)/(n+1) * A(n-1)
Sage] [ A(n) for n in [0..10] ]
```

$[1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796]$

(a) Show that $A(n)$ equals the n-th Catalan number, that is, $A(n) = \frac{1}{n+1}\binom{2n}{n}$.

(b) Show that $A(n) = \binom{2n}{n} - \binom{2n}{n+1}$.

(c) Observe that `A(1).parent()` is the rational numbers, even though 1 is an integer. This is the result of using the division operator `/`. Use the operator `//` to rewrite the function $A(n)$ so that its output is always an integer.

Solution.

(a) The Sage function computes the sequence $A(n)$ defined by

$$A(n) = \frac{2(2n-1)}{n+1} A(n-1), \quad A(0) = 1.$$

It is straightforward to check that this first-order recurrence is also satisfied by, and hence characterizes, the Catalan numbers.

(b) Note that

$$\frac{\binom{2n}{n+1}}{\binom{2n}{n}} = \frac{n!n!}{(n+1)!(n-1)!} = \frac{n}{n+1}.$$

Hence,

$$\binom{2n}{n} - \binom{2n}{n+1} = \left(1 - \frac{n}{n+1}\right)\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n}.$$

(c) Do it! The lesson is that Sage is very careful about which, say, ring an object belongs to. The division of, for instance, two polynomials will produce something that lives in the fraction field of rational functions. This same behaviour is observed here when dividing two integers. If we know that the division results in another integer, we can use `//` instead to avoid passage to the fraction field (in general, `//` and its companion `%` can be used for division with remainder).

```
Sage] (4/2).parent()
```

$\mathbb{Q}$

```
Sage] (4//2).parent()
```

$\mathbb{Z}$

```
Sage] 5//2
```

2

```
Sage] 5%2
```

1

```
Sage] R.<x> = QQ[]
Sage] (x^2-1)/(x-1)
```

$x + 1$

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```
Sage] ((x^2-1)/(x-1)).parent()
```

Frac($\mathbb{Q}[x]$)

```
Sage] ((x^2-1)//(x-1)).parent()
```

$\mathbb{Q}[x]$

```
Sage] (x^2-1)//(x-1)
```

$x+1$

```
Sage] (x^2+1)//(x-1)
```

$x+1$

```
Sage] (x^2+1)%(x-1)
```

2

```
Sage]
```

$\square$

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