Example 137. Python Various integration methods are already implemented in scipy.

```
>>> from scipy import integrate
```

For instance, the following is a way to use Simpson's rule with $n=4$ (so that 5 points are used). The result matches the $\frac{11}{10}$ that we computed ourselves.

```
>>> def f(x):
        return 1/x
>>> xvalues = [1+1/2*i for i in range(5)]
>>> yvalues = [f(x) for x in xvalues]
>>> integrate.simps(yvalues, xvalues)
    1.0999999999999999
```

On the other hand, the following is a convenient way of "general purpose integration", where we only need to specify the end points:

```
>>> integrate.quad(f, 1, 3)
    (1.0986122886681096, 7.555511459798467e-14)
```

The second part of the result is an estimate for the absolute error.

Numerical methods for solving differential equations

The general form of a first-order differential equation (DE) is $y' = f(x, y)$.

Comment. Recall that higher-order differential equations can be written as systems of first-order differential equations: $\boldsymbol{y}' = f(x, \boldsymbol{y})$ in terms of $\boldsymbol{y} = (y_1, y_2, y_3, \ldots)$ where we set $y_1 = y$, $y_2 = y'$, $y_3 = y''$, ...

It therefore is no loss of generality to develop methods for first-order differential equation. While we will focus on the case of a single function $y(x)$, the methods we discuss extend naturally to the case of several functions $\boldsymbol{y}(x) = (y_1(x), y_2(x), \ldots)$.

In order to have a unique solution $y(x)$ that we can numerically approximate, we will add an initial condition. As such, we discuss methods for solving first-order initial value problems (IVPs)

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Comment. Recall from your Differential Equations class that such an IVP is guaranteed to have a unique solution under mild assumptions on $f(x, y)$ (for instance, that $f(x, y)$ is smooth around (x_0, y_0)).

Comment. There would be no loss of generality in only considering initial conditions of the form $y(0) = y_0$. Indeed, suppose the initial condition is $y(x_0) = y_0$. Then, by replacing x by $x + x_0$ in the DE and rewriting the DE in terms of $\tilde{y}(x) = y(x + x_0)$, we obtain an IVP with initial condition $\tilde{y}(0) = y_0$.

Review of the simplest differential equations

Let's start with one of the simplest (and most fundamental) differential equation (DE). It is **first-order** (only a first derivative) and **linear** (with constant coefficients).

Example 138. Solve $y' = 3y$.

Solution. The general solution is $y(x) = Ce^{3x}$. (Note that there is 1 degree of freedom corresponding to the fact that the differential equation is of order 1.)

Check. Indeed, if $y(x) = Ce^{3x}$, then $y'(x) = 3Ce^{3x} = 3y(x)$.

Comment. Recall we can always easily check whether a function solves a differential equation. This means that (although you might be unfamiliar with certain techniques for solving) you can use computer algebra systems to solve differential equations without trust issues.

To describe a unique solution, additional constraints need to be imposed.

Example 139. Solve the **initial value problem** (IVP) $y' = 3y$, $y(0) = 5$.

Solution. This has the unique solution $y(x) = 5e^{3x}$.

The following is a **non-linear** differential equation. In general, such equations are much more complicated than linear ones. We can solve this particular one because it is **separable**.

Example 140. Solve $y' = xy^2$.

Solution. This DE is separable: $\frac{1}{y^2}\mathrm{d}y = x\,\mathrm{d}x$. Integrating, we find $-\frac{1}{y} = \frac{1}{2}x^2 + C$.

Hence, $y = -\frac{1}{\frac{1}{2}x^2 + C} = \frac{2}{D - x^2}$. [Here, $D = -2C$ but that relationship doesn't matter.]

Comment. Note that we did not find the singular solution $y = 0$ (lost when dividing by y^2). We can obtain it from the general solution by letting $D \to \infty$.

Euler's method

Euler's method is a numerical way of approximating the (unique) solution $y(x)$ to the IVP

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Choose a step size $h > 0$. Write $x_n = x_0 + nh$. Our goal is to provide approximations y_n of $y(x_n)$ for $n = 1, 2, \ldots$. Recall that, by Taylor's theorem (Theorem 54), we have

$$y(x + h) = y(x) + y'(x)h + \frac{1}{2}y''(\xi)h^2.$$

Therefore, we can approximate $y(x_1)$ as follows:

$$y(x_1) = y(x_0 + h) \approx y(x_0) + y'(x_0)h \overset{\text{DE}}{=} \underbrace{y(x_0)}_{= y_0} + f(x_0, \underbrace{y(x_0)}_{= y_0})h \quad =: \quad y_1.$$

Here, the **local truncation error** is $O(h^2)$ (note that this is only the error when going from x_0 to x_1). We then likewise approximate

$$y(x_2) = y(x_1 + h) \approx y(x_1) + y'(x_1)h \overset{\text{DE}}{=} \underbrace{y(x_1)}_{\approx y_1} + f(x_1, \underbrace{y(x_1)}_{\approx y_1})h \quad =: \quad y_2.$$

Note that, at this point we don't know the exact value of $y(x_1)$ but instead use our previous approximation y_1 to get y_2. Continuing like that, we have

$$y(x_{n+1}) = y(x_n + h) \quad \approx \quad y(x_n) + \underbrace{y'(x_n)}_{f(x_n, y(x_n))} h \quad \approx \quad y_n + f(x_n, y_n)h \quad =: \quad y_n.$$

Two kinds of errors. There are two different errors involved here: in the first approximation, the error is from truncating the Taylor expansion and we know that this **local truncation error** is $O(h^2)$. On the other hand, in the second approximation, we introduce an error because we use the previous approximation y_n instead of $y(x_n)$.

Suppose that we approximate $y(x)$ on some interval $[x_0, x_{\max}]$ using n steps (so that $x_n = x_{\max}$).

Then the step size is $h = \frac{x_{\max} - x_0}{n}$. We therefore have $n = \frac{x_{\max} - x_0}{h}$ many local truncation errors of size $O(h^2)$.

It is therefore natural to expect that the **global error** is $O(nh^2) = O(h)$.

(Euler's method) The following is an order 1 method for solving IVPs:

$$\begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + f(x_n, y_n)h \end{aligned}$$

Comment. As explained above, being an order 1 method means that Euler's method has a global error that is $O(h)$ (while the local truncation error is $O(h^2)$).

As illustrated above, the error in the approximations y_1, y_2, ... is accumulating (we are using the previous approximation to generate the next one) and we therefore expect the approximations to become worse as we continue (in other words, our approximations of $y(x)$ will be worse as x gets further away from x_0).

Comment. While Euler's method is rarely used in practice, it serves as the foundation for more powerful extensions such as the Runge–Kutta methods.

Example 141. Consider the IVP $\frac{dy}{dx} = (2x - 3y)^2 + \frac{2}{3}$, $y(1) = \frac{1}{3}$.

(a) Approximate the solution $y(x)$ for $x \in [1, 2]$ using Euler's method with 2 steps.

(b) Approximate the solution $y(x)$ for $x \in [1, 2]$ using Euler's method with 3 steps.

(c) Solve this IVP exactly. Compare the values at $x = 2$.

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Solution.

(a) The step size is $h = \frac{2-1}{2} = \frac{1}{2}$. We apply Euler's method with $f(x,y) = (2x-3y)^2 + \frac{2}{3}$:

$$\begin{aligned} x_0 &= 1 & y_0 &= \frac{1}{3} \\ x_1 &= \frac{3}{2} & y_1 &= y_0 + h f(x_0, y_0) = \frac{1}{3} + \frac{1}{2} \cdot \left[\left(2 \cdot 1 - 3 \cdot \frac{1}{3}\right)^2 + \frac{2}{3}\right] = \frac{7}{6} \\ x_2 &= 2 & y_2 &= y_1 + h f(x_1, y_1) = \frac{7}{6} + \frac{1}{2} \cdot \frac{11}{12} = \frac{13}{8} \end{aligned}$$

In particular, the approximation for $y(2)$ is $y_2 = \frac{13}{8} = 1.625$.

(b) The step size is $h = \frac{2-1}{3} = \frac{1}{3}$. We again apply Euler's method with $f(x,y) = (2x-3y)^2 + \frac{2}{3}$:

$$\begin{aligned} x_0 &= 1 & y_0 &= \frac{1}{3} \\ x_1 &= \frac{4}{3} & y_1 &= y_0 + h f(x_0, y_0) = \frac{1}{3} + \frac{1}{3} \cdot \left[\left(2 \cdot 1 - 3 \cdot \frac{1}{3}\right)^2 + \frac{2}{3}\right] = \frac{8}{9} \\ x_2 &= \frac{5}{3} & y_2 &= y_1 + h f(x_1, y_1) = \frac{8}{9} + \frac{1}{3} \cdot \frac{2}{3} = \frac{10}{9} \\ x_3 &= 2 & y_3 &= y_2 + h f(x_2, y_2) = \frac{10}{9} + \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{3} \end{aligned}$$

In particular, the approximation for $y(2)$ is $y_3 = \frac{4}{3} \approx 1.333$.

(c) This IVP can be solved using the substitution $u = 2x - 3y$ followed by separation of variables. We then find that the unique solution of the IVP is $y(x) = \frac{2}{3}x - \frac{1}{3(3x-2)}$.

In particular, the exact value at $x = 2$ is $y(2) = \frac{5}{4} = 1.25$.

We observe that our approximations for $y(2) = 1.25$ improved from 1.625 to 1.333 as we increased the number of steps (equivalently, we decreased the step size h from $\frac{1}{2}$ to $\frac{1}{3}$).

For comparison. With 10 steps (so that $h = \frac{1}{10}$), the approximation improves to $y(2) \approx 1.259$.