Example 125. (review) Suppose that $A\left(\frac{1}{2}\right)=\frac{3}{8}$ and $A\left(\frac{1}{3}\right)=\frac{5}{12}$ are approximations of order 3 of some quantity A^*. What is the approximation we obtain from using Richardson extrapolation?

Solution. Since $A(h)$ is an approximation of order 3, we expect $A(h)\approx A^*+Ch^3$ for some constant C.

Correspondingly, $A\left(\frac{1}{2}\right)\approx A^*+\frac{1}{8}C$ and $A\left(\frac{1}{3}\right)\approx A^*+\frac{1}{27}C$.

Hence, $27A\left(\frac{1}{3}\right)-8A\left(\frac{1}{2}\right)\approx(27-8)A^*=19A^*$. To get an approximation of A^*, we need to divide by 19.

The Richardson extrapolation is $\frac{27}{19}A\left(\frac{1}{3}\right)-\frac{8}{19}A\left(\frac{1}{2}\right)=\frac{27}{19}\frac{5}{12}-\frac{8}{19}\frac{3}{8}=\frac{33}{76}$.

Example 126. Apply Richardson extrapolation to $f'(x)\approx\frac{1}{h}[f(x+h)-f(x)]$.

Solution. We have seen in Example 114 that the approximation $A(h)=\frac{1}{h}[f(x+h)-f(x)]$ is of order 1. Hence,

$$\begin{aligned}\frac{2A(h)-A(2h)}{2-1} &= \frac{2}{h}[f(x+h)-f(x)]-\frac{1}{2h}[f(x+2h)-f(x)]\\ &= \frac{1}{2h}[-f(x+2h)+4f(x+h)-3f(x)]\end{aligned}$$

is an approximation of higher order (we expect it to be of order 2).

Indeed, this approximation of $f'(x)$ is the same as what we obtained in Example 119 when applying polynomial interpolation to $f(x)$, $f(x+h)$, $f(x+2h)$. As observed there, the error is $-\frac{1}{3}f'''(x)h^2+O(h^3)$ showing that this is indeed an approximation of order 2.

Example 127. Apply Richardson extrapolation to $f'(x)\approx\frac{1}{2h}[f(x+h)-f(x-h)]$.

Solution. We have seen in Example 115 that the approximation $A(h)=\frac{1}{2h}[f(x+h)-f(x-h)]$ is of order 2. Hence,

$$\begin{aligned}\frac{2^2A(h)-A(2h)}{2^2-1} &= \frac{2}{3h}[f(x+h)-f(x-h)]-\frac{1}{12h}[f(x+2h)-f(x-2h)]\\ &= \frac{1}{12h}[-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)]\end{aligned}$$

is an approximation of $f'(x)$ of higher order. With some more work (do it!), we find that the error is $-\frac{1}{30}f^{(5)}(x)h^4+O(h^6)$ so that this is an approximation of order 4.

Comment. Note that the above approximations don't change when h is replaced by $-h$ (note the factor of $1/h$). In other words, the approximations are even functions in h. Consequently, their Taylor expansion in h will only have even powers of h. That's the reason why the order of the Richardson extrapolation is 4 rather than order 3 which is what one would otherwise expect when extrapolating an order 2 formula.

Example 128. Apply Richardson extrapolation to $f''(x)\approx\frac{1}{h^2}[f(x+h)-2f(x)+f(x-h)]$.

Solution. We have seen in Example 118 that the approximation $A(h)=\frac{1}{h^2}[f(x+h)-2f(x)+f(x-h)]$ is of order 2. Hence,

$$\frac{2^2A(h)-A(2h)}{2^2-1}=\frac{1}{12h^2}[-f(x+2h)+16f(x+h)-30f(x)+16f(x-h)-f(x-2h)]$$

is an approximation of $f''(x)$ of higher order. With some more work, we find that the error is $-\frac{1}{90}f^{(6)}(x)h^4+O(h^6)$ so that this is an approximation of order 4.

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In the previous example, we combined $A(h)$ and $A(2h)$ to obtain an approximation of higher order. There is nothing special about $2h$. We can likewise combine $A(h_1)$ and $A(h_2)$ for any h_1, h_2.

Example 129. (homework) $A(h) = \frac{1}{h^2}[f(x+h) - 2f(x) + f(x-h)]$ is an order 2 approximation of $f''(x)$. Apply Richardson extrapolation to $A(h)$ and $A\left(\frac{3}{2}h\right)$ to obtain an approximation of $f''(x)$ of higher order.

Solution. Since $A(h)$ is an approximation of order 2, we expect $A(h) \approx A^* + Ch^2$ for some constant C. Correspondingly, $A\left(\frac{3}{2}h\right) \approx A^* + \frac{9}{4}Ch^2$. Hence, $\frac{9}{4}A(h) - A\left(\frac{3}{2}h\right) \approx \left(\frac{9}{4} - 1\right)A^* = \frac{5}{4}A^*$.

The Richardson extrapolation of $A(h)$ and $A\left(\frac{3}{2}h\right)$ therefore is:

$$\frac{\frac{9}{4}A(h) - A\left(\frac{3}{2}h\right)}{\frac{5}{4}} = \frac{1}{45h^2}\left[-16f\left(x + \frac{3}{2}h\right) + 81f(x+h) - 130f(x) + 81f(x-h) - 16f\left(x - \frac{3}{2}h\right)\right]$$

This is an approximation of $f''(x)$ of higher order. With some more work, we find that the error is $-\frac{1}{160}f^{(6)}(x)h^4 + O(h^6)$ so that this is an approximation of order 4.

Example 130. Python Let us now repeat Example 121 with the formula

$$f'(x) \approx \frac{1}{12h}[-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)]$$

that we obtained in Example 127.

```
>>> def central_difference_richardson(f, x, h):
        return (-f(x+2*h)+8*f(x+h)-8*f(x-h)+f(x-2*h))/(12*h)
```

Let us again approximate $f'(1) = 2\ln(2) \approx 1.386$ for $f(x) = 2^x$ at $x = 1$.

```
>>> [central_difference_richardson(f, 1, 10**-n) for n in range(5)]

[1.375, 1.3862932938249581, 1.3862943610132332, 1.3862943611198109, 1.3862943611187004]
```

Does the error behave as expected?

```
>>> [central_difference_richardson(f, 1, 10**-n) - 2*log(2) for n in range(6)]

[-0.011294361119890572, -1.0672949324330716e-06, -1.0665734961889939e-10, -
7.971401316808624e-14, -1.1901590823981678e-12, 1.5093037930569153e-11]
```

We noted in Example 127 that the approximation is of order 4. Indeed, we can see how the error decreases roughly by $1/10^4$ initially, as expected.

Moreover, we are able to obtain a much better numerical estimate compared to Example 121: this time, our best approximation has error $7.97 \cdot 10^{-14}$, which is decently close to the machine precision of $\varepsilon \approx 2^{-52} \approx 2.2 \cdot 10^{-16}$. This is because the effect of rounding errors becomes devastating as h becomes very small. Using a high-order approximation, we are often able to avoid having to work with very small h.

Indeed, note how we got the best approximation with $h = 10^{-3}$ (whereas we previously needed to a much smaller h for the best approximations).