19 Continued fractions

Definition 164. A **continued fraction** is a fraction of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}},$$

with $a_1, a_2, \ldots$ positive. Written as $[a_0; a_1, a_2, \ldots]$.

Called **simple** if all the a_i are integers.

Example 165. Express $\frac{5}{3}$ as a simple continued fraction.

Solution. $\frac{5}{3} = 1 + \frac{2}{3} = 1 + \frac{1}{3/2} = 1 + \frac{1}{1 + \frac{1}{2}} = [1; 1, 2]$

Writing the final 2 as $1 + \frac{1}{1}$, we also have $\frac{5}{3} = 1 + \frac{1}{1 + \frac{1}{2}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}} = [1; 1, 1, 1]$.

More generally. If $a_n > 1$, we always have $[a_0; a_1, a_2, \ldots, a_n] = [a_0; a_1, a_2, \ldots, a_n - 1, 1]$.

Comment. Apart from these two variations, the simple continued fraction for $\frac{5}{3}$ is unique.

Note that we are used to a similar ambiguity when dealing with terminating decimal expansions: for instance, $1.25000000\ldots = 1.24999999\ldots$

A slight variation. It follows from the above that $\frac{3}{5} = 0 + \frac{1}{5/3} = 0 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}} = [0; 1, 1, 2]$.

More generally, we always have that, if $x = [a_0; a_1, a_2, \ldots]$ with $a_0 > 0$, then $\frac{1}{x} = [0; a_0, a_1, a_2, \ldots]$.

Example 166. Express $\frac{43}{19}$ as a simple continued fraction.

Solution. $\frac{43}{19} = 2 + \frac{5}{19} = 2 + \frac{1}{19/5} = 2 + \frac{1}{3 + \frac{4}{5}} = 2 + \frac{1}{3 + \frac{1}{5/4}} = 2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{4}}} = [2; 3, 1, 4]$

Again, also, $\frac{43}{19} = [2; 3, 1, 4] = \left[2; 3, 1, 3 + \frac{1}{1}\right] = [2; 3, 1, 3, 1]$.

Super important observation. We have done this computation before (in a different guise)!

By the Euclidean algorithm: $43 = \boxed{2} \cdot 19 + 5$, $19 = \boxed{3} \cdot 5 + 4$, $5 = \boxed{1} \cdot 4 + 1$, $4 = \boxed{4} \cdot 1 + 0$.

Example 167. Evaluate $[2; 3]$, $[2; 3, 4]$, and $[2; 3, 4, 5]$.

Solution.

$[2; 3] = 2 + \frac{1}{3} = \frac{7}{3} \approx 2.333$

$[2; 3, 4] = 2 + \frac{1}{3 + \frac{1}{4}} = 2 + \frac{4}{13} = \frac{30}{13} \approx 2.308$

$[2; 3, 4, 5] = 2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{5}}} = 2 + \frac{1}{3 + \frac{5}{21}} = 2 + \frac{21}{68} = \frac{157}{68} \approx 2.309$

Definition 168. The **convergents** C_k of $[a_0; a_1, a_2, \ldots]$ are the truncated continued fractions $C_0 = a_0$, $C_1 = [a_0; a_1]$, $C_2 = [a_0; a_1, a_2]$, ..., $C_k = [a_0; a_1, a_2, \ldots, a_k]$, ...

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Theorem 169. The convergents C_k of a simple continued fraction $[a_0; a_1, a_2, \ldots]$ always converge to a value x in the following alternating fashion

$$C_0 < C_2 < C_4 < \cdots \quad x \quad \cdots < C_5 < C_3 < C_1.$$

We simply write $x = [a_0; a_1, a_2, \ldots]$ for that value.

If the continued fraction is finite, that is $x = [a_0; a_1, a_2, \ldots, a_n]$, then we only have the convergents $C_0, C_1, \ldots, C_n$ and $C_n = x$.

Proof. From

$$C_0 = a_0, \quad x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}, \quad C_1 = a_0 + \frac{1}{a_1},$$

we see that C_0 is less than x, as well as less than all other convergents (because all of these equal a_0 plus something positive). Similarly, C_1 is larger than x, as well as larger than all other convergents.

The full claim then follows by, likewise, looking at $[a_1; a_2, a_3 \ldots]$ in place of $[a_0; a_1, a_2, \ldots]$.

See Theorem 15.4 in our book for full details. □

Theorem 170. (representing a real number as a simple continued fraction)

- An irrational number x has a unique representation as a simple continued fraction. This continued fraction is infinite.
- A rational number x has exactly two representations as a simple continued fraction. Both are finite (one ends in a 1 and the other doesn't).

Proof. Let x be a positive real number. Let us think about how a continued fraction for x has to look like.

[The argument for negative x is essentially the same. For negative x, a_0 will be negative but the remainder and the other digits are positive.]

As in Theorem 169, we have $C_0 \leqslant x \leqslant C_1$ where $C_0 = a_0$ and $C_1 = a_0 + \frac{1}{a_1} \leqslant a_0 + 1$.

Hence, $a_0 \leqslant x \leqslant a_0 + 1$ which means that a_0 has to be the integer $a_0 = \lfloor x \rfloor$.

(unless) Well, unless x is an integer itself, in which case we have the two possibilities $a_0 = x$ or $a_0 = x - 1$. But in that special case, we are done: the continued fraction for x is finite and there are exactly the two representations $x = [x]$ and $x = [x - 1; 1]$.

So, $x = a_0 + \frac{1}{y}$ with $y = \frac{1}{x - a_0} > 0$, and the continued fraction for x is $x = [a_0; b_0, b_1, \ldots]$ provided that y has the continued fraction $y = [b_0; b_1, \ldots]$. We now repeat our argument, starting with the positive real number y (so that $b_0 = \lfloor y \rfloor$, ...).

There are two possibilities:

- The process stops along the way because the number we are looking at is an integer (the "unless" case). In that case, we get exactly two finite simple continued fractions for x (one of which ends in 1).

  This happens if and only if x is rational (from the Euclidean algorithm we know that every rational number has a finite simple continued fraction; conversely, a finite simple continued fraction necessarily represents a rational number).

- The process continues indefinitely. In that case, we get a (unique) infinite simple continued fraction for x. (By Theorem 169, this continued fraction indeed converges to x.) □

Review. Euler's number $e = 2.71828182846\ldots$ and its significance (differential equations, compound interest)

Example 171. Express $\frac{55}{24}$ as a simple continued fraction.

Solution. By the Euclidean algorithm: $55 = \boxed{2} \cdot 24 + 7$, $24 = \boxed{3} \cdot 7 + 3$, $7 = \boxed{2} \cdot 3 + 1$, $3 = \boxed{3} \cdot 1 + 0$.
Hence, $\frac{55}{24} = [2; 3, 2, 3]$.

Example 172. Determine the first few digits of the simple continued fraction of e.

Solution. $e = \boxed{2}.71828182846\ldots$

$e = 2 + \frac{1}{1/0.7182\ldots} = [2; a_1, a_2, \ldots]$ where $[a_1; a_2, \ldots] = 1/0.7182\ldots = \boxed{1}.3922\ldots$

$1/0.3922\ldots = \boxed{2}.5496\ldots$, $1/0.5496\ldots = \boxed{1}.8194\ldots$, $1/0.8194\ldots = \boxed{1}.2205\ldots$, $1/0.2205\ldots = \boxed{4}.5356\ldots$

Hence, $e = [2; 1, 2, 1, 1, 4, \ldots]$. Computing further, $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$ and the pattern continues.

Note. Assuming that the pattern does continue, this proves that e is irrational!

Example 173.

(a) Evaluate the first 4 convergents of $[2; 3, 2, 3, 2, \ldots]$ (and then, using the next result, compute 3 more convergents).

(b) Which number is represented by $[2; 3, 2, 3, 2, \ldots]$?

Solution.

(a) $C_0 = 2$

$C_1 = [2; 3] = 2 + \frac{1}{3} = \frac{7}{3} \approx 2.333$

$C_2 = [2; 3, 2] = 2 + \frac{1}{3 + \frac{1}{2}} = 2 + \frac{2}{7} = \frac{16}{7} \approx 2.286$

$C_3 = [2; 3, 2, 3] = 2 + \frac{1}{3 + \frac{1}{2 + \frac{1}{3}}} = \frac{55}{24} \approx 2.292$

Using the next result, we compute the convergents $C_n = \frac{p_n}{q_n}$ as follows:

n	-2	-1	0	1	2	3	4	5	6
a_n			2	3	2	3	2	3	2
p_n	0	1	2	7	16	55	126	433	992
q_n	1	0	1	3	7	24	55	189	433
C_n			$\frac{2}{1}$	$\frac{7}{3}$	$\frac{16}{7}$	$\frac{55}{24}$	$\frac{126}{55}$	$\frac{433}{189}$	$\frac{992}{433}$

(b) Write $x = [2; 3, 2, 3, 2, \ldots]$. Then, $x = 2 + \frac{1}{3 + \frac{1}{2 + \frac{1}{3 + \ldots}}} = 2 + \frac{1}{3 + \frac{1}{x}}$.

The equation $x = 2 + \frac{1}{3 + \frac{1}{x}}$ simplifies to $x - 2 = \frac{x}{3x + 1}$.

Further (note that, clearly $x \neq -\frac{1}{3}$ so that $3x + 1 \neq 0$) simplifies to $(x - 2)(3x + 1) = x$ or $3x^2 - 6x - 2 = 0$, which has the solutions $x = \frac{6 \pm \sqrt{36 + 24}}{6} = 1 \pm \sqrt{\frac{5}{3}}$.

Since $1 + \sqrt{\frac{5}{3}} \approx 2.291$ and $1 - \sqrt{\frac{5}{3}} \approx -0.291$, we conclude that $[2; 3, 2, 3, 2, \ldots] = 1 + \sqrt{\frac{5}{3}}$.

Advanced comment. The fractions $\frac{p_n}{q_n}$ are always reduced! Can you see how to conclude that $\gcd(p_n, q_n) = 1$ from the relation $p_n q_{n-1} - p_{n-1} q_n = (-1)^n$ (which can be proved by induction)?

We can see this relation quite nicely in the above table because $p_n q_{n-1} - p_{n-1} q_n$ is a 2×2 determinant taken from the rows containing p_n and q_n:

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1, \quad \begin{vmatrix} 2 & 7 \\ 1 & 3 \end{vmatrix} = -1, \quad \begin{vmatrix} 7 & 16 \\ 3 & 7 \end{vmatrix} = 1, \quad \begin{vmatrix} 16 & 55 \\ 7 & 24 \end{vmatrix} = -1, \quad \ldots$$

Theorem 174. The kth convergent of the continued fraction $[a_0; a_1, a_2, \ldots]$ is

$$C_k = \frac{p_k}{q_k},$$

where p_k and q_k are characterized by

$$\begin{array}{lcl} p_k = a_k p_{k-1} + p_{k-2} & \text{and} & q_k = a_k q_{k-1} + q_{k-2} \\ \text{with } p_{-2} = 0, \quad p_{-1} = 1 & & \text{with } q_{-2} = 1, \quad q_{-1} = 0 \end{array}.$$

Proof. We will prove the claim by induction on k. (More on that technique next time!)

First, we check the two base cases $k = 0$, $k = 1$ directly: $C_0 = a_0$ and $C_1 = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1}$. In other words, $p_0 = a_0$, $q_0 = 1$ and $p_1 = a_0 a_1 + 1$, $q_1 = a_1$. This matches with the values from the recursion.

Next, we assume that the theorem is true for $k = 0, 1, \ldots, n$. In particular,

$$C_n = [a_0; a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}},$$

for all values of $a_0, a_1, \ldots, a_n$. Note that $C_{n+1} = [a_0; a_1, a_2, \ldots, a_n, a_{n+1}] = \left[a_0; a_1, a_2, \ldots, a_n + \frac{1}{a_{n+1}}\right]$. Replacing a_n with $a_n + \frac{1}{a_{n+1}}$, we therefore obtain

$$\begin{aligned} C_{n+1} = \left[a_0; a_1, a_2, \ldots, a_n + \frac{1}{a_{n+1}}\right] &= \frac{\left(a_n + \frac{1}{a_{n+1}}\right) p_{n-1} + p_{n-2}}{\left(a_n + \frac{1}{a_{n+1}}\right) q_{n-1} + q_{n-2}} \\ &= \frac{(a_n a_{n+1} + 1) p_{n-1} + a_{n+1} p_{n-2}}{(a_n a_{n+1} + 1) q_{n-1} + a_{n+1} q_{n-2}} \\ &= \frac{a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1}}{a_{n+1}(a_n q_{n-1} + q_{n-2}) + q_{n-1}} \\ &= \frac{a_{n+1} p_n + p_{n-1}}{a_{n+1} q_n + q_{n-1}} = \frac{p_{n+1}}{q_{n+1}}. \end{aligned}$$

The claim now follows by induction. □

Example 175. Determine $[1; 1, 1, 1, \ldots]$ as well as its first 6 convergents.

Solution. The first few convergents are $C_0 = 1$, $C_1 = [1; 1] = 2$, $C_2 = [1; 1, 1] = 1 + \frac{1}{1 + \frac{1}{1}} = \frac{3}{2}$.

Since this starts getting tedious, we instead compute the convergents $C_n = \frac{p_n}{q_n}$ recursively:

n	-2	-1	0	1	2	3	4	5	6
a_n			1	1	1	1	1	1	1
p_n	0	1	1	2	3	5	8	13	21
q_n	1	0	1	1	2	3	5	8	13
C_n			1	2	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{8}{5}$	$\frac{13}{8}$	$\frac{21}{13}$

Note that the C_n are quotients of Fibonacci numbers ($F_0 = 0, F_1 = 1, F_2 = 1, \ldots$)! To be precise, $C_n = \frac{F_{n+2}}{F_{n+1}}$.

Next, let's determine $x = [1; 1, 1, 1, \ldots]$ by observing that $x = 1 + \frac{1}{1 + \frac{1}{1 + \ldots}} = 1 + \frac{1}{x}$.

The equation $x = 1 + \frac{1}{x}$ simplifies to $x^2 - x - 1 = 0$, which has the solutions $x = \frac{1 \pm \sqrt{5}}{2}$.

Since $\frac{1 - \sqrt{5}}{2}$ is negative (while x is between $C_0 = 1$ and $C_1 = 2$), we conclude $[1; 1, 1, 1, \ldots] = \frac{1 + \sqrt{5}}{2} \approx 1.618$.

This is the **golden ratio** φ.

Comment. Note that we have shown, in particular, $\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \varphi \approx 1.618$.

Comment. As noticed in the previous example, the fractions $\frac{p_n}{q_n} = \frac{F_{n+2}}{F_{n+1}}$ are always reduced. In other words, $\gcd(F_n, F_{n+1}) = 1$. Moreover, $p_n q_{n-1} - p_{n-1} q_n = (-1)^n$ implies that $F_n^2 - F_{n-1} F_{n+1} = (-1)^{n+1}$.

Example 176. Determine the first few digits of the simple continued fraction of π, as well as the first few convergents.

Solution. $\pi = \boxed{3}.14159265359...$, $1/0.14159... = \boxed{7}.06251...$, $1/0.06251... = \boxed{15}.99659...$, $1/0.99659... = \boxed{1}.00341...$, $1/0.00341... = \boxed{292}.63459...$

Continuing this way, we find $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, ...]$.

Since π is irrational, this is an infinite continued fraction. No pattern in this fraction is known.

We compute the convergents $C_n = \frac{p_n}{q_n}$ as follows:

n	-2	-1	0	1	2	3	4	5	6
a_n			3	7	15	1	292	1	1
p_n	0	1	3	22	333	355	$103,993$	$...$	$...$
q_n	1	0	1	7	106	113	$33,102$	$...$	$...$
C_n			3	$\frac{22}{7}$	$\frac{333}{106}$	$\frac{355}{113}$	$\frac{103,993}{33,102}$	$...$	$...$

Comment. For $n \geqslant 1$, each approximation $x \approx \frac{p_n}{q_n}$ is best possible in the sense that it is better than any other approximation $\frac{a}{b}$ with $b \leqslant q_n$. In other words, if $\left|x - \frac{a}{b}\right| < \left|x - \frac{p_n}{q_n}\right|$, then $b > q_n$.

Comment. Because of this, it is natural to expect that the approximations $\frac{22}{7}$ and $\frac{355}{113}$ are particularly good, because they are followed by much "bigger" fractions.

Indeed, $\frac{22}{7} = \boxed{3.14}28...$ and $\frac{355}{113} = \boxed{3.141592}92...$ are very good approximations to π.

Comment. It is known that π is irrational, so that the above "wild" continued fraction will go on forever.

Embarrassingly, we do not know whether, for instance, $e + \pi = 5.85987448205...$ is irrational.

$e + \pi = [5; 1, 6, 7, 3, 21, 2, 1, 2, 2, 1, 1, 2, 3, 3, 2, 5, 2, 1, 1, ...]$

All evidence points to it being irrational, but nobody has a proof. (In particular, we cannot be sure that this continued fraction goes on forever.)

Comment. Among other approximations to π, Ramanujan suggested $\pi \approx \left(97.5 - \frac{1}{11}\right)^{1/4}$. Can you explain how one might discover this?

[Hint: Compute the continued fraction of π^4!]

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20 Basic proof techniques

20.1 Proofs by contradiction

Example 177. (again) $\sqrt{5}$ is not rational.

Proof. Assume (for contradiction) that we can write $\sqrt{5} = \frac{n}{m}$ with $n, m \in \mathbb{N}$. By canceling common factors, we can ensure that this fraction is reduced.

Then $5m^2 = n^2$, from which we conclude that n is divisible by 5. Write $n = 5k$ for some $k \in \mathbb{N}$. Then $5m^2 = (5k)^2$ implies that $m^2 = 5k^2$. Hence, m is also divisible by 5. This contradicts the fact that the fraction n/m is reduced. Hence, our initial assumption must have been wrong. □

Variations. Does the same proof apply to, say, $\sqrt{7}$?

Which step of the proof fails for $\sqrt{9}$?

Comment. We showed earlier that $[1; 1, 1, 1, \ldots] = \frac{1+\sqrt{5}}{2}$. Since this is an infinite continued fraction, this proves that $\frac{1+\sqrt{5}}{2}$ is irrational. Consequently, $\sqrt{5}$ is irrational as well.

20.2 A famous example of a direct proof

Example 178. (Gauss) $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$

Proof. Write $s(n) = 1 + 2 + \ldots + n$.

$2s(n) = (1 + 2 + \ldots + n) + (n + (n-1) + \ldots + 1) = (1+n) + (2+n-1) + \ldots + (n+1) = n \cdot (n+1)$. Done! □

Anecdote. 9 year old Gauss (1777-1855) and his classmates were tasked to add the numbers 1 to 100 (and not bother their teacher while doing so). Gauss was not writing much on his slate... just the final answer: 5050.

20.3 Proofs by induction

(induction) To prove that $\mathrm{CLAIM}(n)$ is true for all integers $n \geqslant n_0$, it suffices to show:

- **(base case)** $\mathrm{CLAIM}(n_0)$ is true.
- **(induction step)** If $\mathrm{CLAIM}(n)$ is true for some n, then $\mathrm{CLAIM}(n+1)$ is true as well.

Why does this work? By the base case, $\mathrm{CLAIM}(n_0)$ is true. Thus, by the induction step, $\mathrm{CLAIM}(n_0+1)$ is true. Applying the induction step again shows that $\mathrm{CLAIM}(n_0+2)$ is true, ...

Comment. In the induction step, we may even assume that $\mathrm{CLAIM}(n_0), \mathrm{CLAIM}(n_0+1), \ldots, \mathrm{CLAIM}(n)$ are all true. This is sometimes referred to as **strong induction**.

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Example 179. (Gauss, again) For all integers $n \geqslant 1$, $1+2+\ldots+n=\frac{n(n+1)}{2}$.

Proof. Again, write $s(n)=1+2+\ldots+n$.
$\mathrm{CLAIM}(n)$ is that $s(n)=\frac{n(n+1)}{2}$.

- **(base case)** $\mathrm{CLAIM}(1)$ is that $s(1)=\frac{1(1+1)}{2}=1$. That's true.
- **(induction step)** Assume that $\mathrm{CLAIM}(n)$ is true (the **induction hypothesis**) for some fixed n.

$$s(n+1)=s(n)+(n+1)=\underbrace{\frac{n(n+1)}{2}}_{\substack{\text{this is where we use}\\ \text{the induction hypothesis}}}+(n+1)=\frac{(n+1)(n+2)}{2}$$

  This shows that $\mathrm{CLAIM}(n+1)$ is true as well.

By induction, the formula is therefore true for all integers $n \geqslant 1$. □

Comment. The claim is also true for $n=0$ (if we interpret the left-hand side correctly).

Example 180. Induction is not only a proof technique but also a common way to define things.

- The **factorial** $n!$ can be defined inductively (i.e. recursively) by

$$0!=1, \quad (n+1)!=n!\cdot(n+1).$$

  Comment. This may not seem impressive, because we can "spell out" $n!=1\cdot 2\cdot 3\cdots(n-1)n$ directly.

- The **Fibonacci numbers** F_n are defined inductively (i.e. recursively) by

$$F_0=0, \quad F_1=1, \quad F_{n+1}=F_n+F_{n-1}.$$

  Getting a feeling. $F_2=F_1+F_0=1$, $F_3=F_2+F_1=2$, $F_4=3$, $F_5=5$, $F_6=8$, $F_7=13$, ...

  Comment. Though not at all obvious, there is a way to compute F_n directly. Let $\varphi=\frac{1+\sqrt{5}}{2}\approx 1.618$. Then $F_n=\lfloor\varphi^n/\sqrt{5}\rceil$. Try it! For instance, $\varphi^{10}/\sqrt{5}\approx 55.0036$. That seems like magic at first. But it is the beginning of a general theory (look up, for instance, Binet's formula and C-finite sequences). Also, recall that we observed that F_{n+1}/F_n are the convergents of the continued fraction for φ.

Example 181. We are interested in the sums $s(n)=1+2+4+\ldots+2^n$.

Getting a feeling. $s(1)=1+2=3$, $s(2)=1+2+4=7$, $s(3)=1+2+4+8=15$, $s(4)=31$

Conjecture. $s(n)=2^{n+1}-1$.

Proof by induction. The statement we want to prove by induction is: $s(n)=2^{n+1}-1$ for all integers $n\geqslant 1$.

- **(base case)** $s(1)=1=2^{1+1}-1$ verifies that the claim is true for $n=1$.
- **(induction step)** Assume that $s(n)=2^{n+1}-1$ is true for some fixed n.
  We need to show that $s(n+1)=2^{n+2}-1$.
  Using the induction hypothesis, $s(n+1)=s(n)+2^{n+1}\overset{\mathrm{IH}}{=}(2^{n+1}-1)+2^{n+1}=2^{n+2}-1$. QED!

Direct proof. $2s(n)=2(1+2+4+\ldots+2^n)=2+4+\ldots+2^{n+1}=s(n)-1+2^{n+1}$. Hence, $s(n)=2^{n+1}-1$.

Example 182. (extra) Can we generalize the previous example by replacing 2 with x?
That is, we are now interested in the sums $s(n) = 1 + x + x^2 + \ldots + x^n$.

Mimic previous direct approach. $x s(n) = x(1 + x + x^2 + \ldots + x^n) = x + x^2 + \ldots + x^{n+1} = s(n) - 1 + x^{n+1}$.
Hence, $(x-1)s(n) = x^{n+1} - 1$, and we have found:

$$1 + x + x^2 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1} \qquad \textbf{(geometric sum)}$$

Sigma notation. Instead of $1 + x + x^2 + \ldots + x^n$ we will begin to write $\sum_{k=0}^{n} x^k$.

Geometric series. We can let $n \to \infty$ to get $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, provided that $|x| < 1$.

Example 183. (homework) Prove the formula for geometric sums using induction.

Example 184. (sum of squares) For all integers $n \geqslant 1$, $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof. Write $t(n) = 1^2 + 2^2 + \ldots + n^2$.
We use induction on the claim $t(n) = \frac{n(n+1)(2n+1)}{6}$.

- The base case $(n = 1)$ is that $t(1) = 1$. That's true.
- For the inductive step, assume the formula holds for some value of n.
  We need to show the formula also holds for $n + 1$.

$$\begin{aligned} t(n+1) &= t(n) + (n+1)^2 \\ \text{(using the induction hypothesis)} &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{(n+1)}{6}[2n^2 + n + 6n + 6] \\ &= \frac{(n+1)}{6}(n+2)(2n+3) \end{aligned}$$

  This shows that the formula also holds for $n + 1$.

By induction, the formula is true for all integers $n \geqslant 1$. □

Example 185. Observe the following connection with our sums and integrals from calculus:

- $\int_0^n x \mathrm{d}x = \frac{n^2}{2}$ versus $\sum_{x=0}^{n} x = 1 + 2 + \ldots + n = \frac{n(n+1)}{2} = \frac{n^2}{2} + \text{lower order terms}$
- $\int_0^n x^2 \mathrm{d}x = \frac{n^3}{3}$ versus $\sum_{x=0}^{n} x^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \text{lower order terms}$
- $\int_0^n x^3 \mathrm{d}x = \frac{n^4}{4}$ versus $\sum_{x=0}^{n} x^3 = 1^3 + 2^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^4}{4} + \text{lower order terms}$

The connection makes sense: the integrals give areas below curves, and the sums are approximations to these areas (rectangles of width 1).

Example 186. (Riemann hypothesis) The **Riemann zeta function** $\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}$ converges (for real s) if and only if $s > 1$.

The divergent series $\zeta(1)$ is the harmonic series, and $\zeta(p)$ is often called a p-series in Calculus II.

Comment. Euler achieved worldwide fame by discovering and proving that $\zeta(2) = \frac{\pi^2}{6}$ (and similar formulas for $\zeta(4), \zeta(6), \ldots$).

For complex values of $s \neq 1$, there is a unique way to "analytically continue" this function. It is then "easy" to see that $\zeta(-2) = 0$, $\zeta(-4) = 0$, $\ldots$. The **Riemann hypothesis** claims that all other zeroes of $\zeta(s)$ lie on the line $s = \frac{1}{2} + a\sqrt{-1}$ ($a \in \mathbb{R}$). A proof of this conjecture (checked for the first 10,000,000,000,000 zeroes) is worth $1,000,000.

http://www.claymath.org/millennium-problems/riemann-hypothesis

The connection to primes. Here's a vague indication that $\zeta(s)$ is intimately connected to prime numbers:

$$\begin{aligned}\zeta(s) &= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \ldots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \ldots\right)\left(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \ldots\right)\cdots \\ &= \frac{1}{1 - 2^{-s}} \frac{1}{1 - 3^{-s}} \frac{1}{1 - 5^{-s}} \cdots \\ &= \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}\end{aligned}$$

This infinite product is called the Euler product for the zeta function. If the Riemann hypothesis was true, then we would be better able to estimate the number $\pi(x)$ of primes $p \leqslant x$.

More generally, certain statements about the zeta function can be translated to statements about primes. For instance, the (non-obvious!) fact that $\zeta(s)$ has no zeros for $\operatorname{Re} s = 1$ implies the prime number theorem that we discussed earlier.

http://www-users.math.umn.edu/~garrett/m/v/pnt.pdf