Homework #2

MATH 311 — Intro to Number Theory
due in class on Thursday, Sep 8

Please print your name:

These problems are not suited to be done last minute!
Also, if you start early, you can consult with me if you should get stuck.

Problem 1.

(a) Write down the first 6 rows of the Pascal triangle.

(b) Expand $(x+y)^6$.

(c) For each row in Pascal's triangle, compute the sum of all entries in that row. Conjecture a formula.

(d) Prove that formula using the binomial theorem.

Solution.

(a)
$$\begin{array}{ccccccccccccc}
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

(b) $(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

(c) Let $s(n)$ be the sum of the entries in row n.

Then: $s(1) = 1+1 = 2$, $s(2) = 1+2+1 = 4$, $s(3) = 8$, $s(4) = 16$, $s(5) = 32$, $s(6) = 64$

Conjecture. $s(n) = 2^n$

(d) Note that

$$s(n) = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}.$$

On the other hand, the binomial theorem states that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

This reduces to $s(n)$ if we set $x = 1$ and $y = 1$. Hence,

$$s(n) = (1+1)^n = 2^n,$$

just as we had conjectured. $\square$

Problem 2. In class, we gave a combinatorial argument showing that

$$\binom{n+1}{3} = \binom{n}{2} + \binom{n-1}{2} + \ldots + \binom{3}{2} + \binom{2}{2}.$$

Armin Straub
straub@southalabama.edu

Prove that formula using induction.

Solution. Write $s(n) = \binom{n}{2} + \binom{n-1}{2} + \ldots + \binom{3}{2} + \binom{2}{2}$.
We use induction on the claim that $s(n) = \binom{n+1}{3}$.

- The base case $(n=2)$ is that $s(2) = \binom{2}{2}$ equals $\binom{2+1}{3}$. That's true.

- For the inductive step, assume the formula holds for some value of n.

  We need to show the formula also holds for $n+1$.

$$\begin{aligned} s(n+1) &= s(n) + \binom{n+1}{2} \\ \text{(using the induction hypothesis)} &= \binom{n+1}{3} + \binom{n+1}{2} \\ \text{(Pascal's rule)} &= \binom{n+2}{3} \end{aligned}$$

  This shows that the formula also holds for $n+1$.

By induction, the formula is true for all integers $n \geqslant 0$.

Comment. It follows from Pascal's rule that $\binom{n+1}{3} + \binom{n+1}{2} = \binom{n+2}{3}$.

[If you want to test your understanding: can you quickly justify this formula using the interpretation of $\binom{n}{k}$ as choosing k from n objects?]

If you didn't see that, you can also write everything as polynomials in n:

$$\binom{n+1}{3} + \binom{n+1}{2} = \frac{(n+1)n(n-1)}{3!} + \frac{(n+1)n}{2} = \frac{(n+1)n}{6}[(n-1)+3] = \frac{(n+2)(n+1)n}{6} = \binom{n+2}{3} \qquad \square$$

Problem 3. Which are the possible remainders that the square of an integer leaves upon division by 5?

Solution. Since we are dividing by 5, it is natural to distinguish the following five cases:

- If $x = 5q$, then $x^2 = 25q^2$ leaves remainder 0.
- If $x = 5q+1$, then $x^2 = 25q^2 + 10q + 1$ leaves remainder 1.
- If $x = 5q+2$, then $x^2 = 25q^2 + 20q + 4$ leaves remainder 4.
- If $x = 5q+3$, then $x^2 = 25q^2 + 30q + 9$ leaves remainder 4.
- If $x = 5q+4$, then $x^2 = 25q^2 + 40q + 16$ leaves remainder 1.

In summary, the only possible remainders are $0, 1, 4$. (Remainders $2, 3$ are not possible.) $\square$

Problem 4.

(a) Prove or disprove: for any integer x, one of the integers x, $x+2$, $x+4$ is divisible by 3.

(b) Prove or disprove: for any integer x, one of the integers x, $x+2$, $x+8$ is divisible by 3.

(c) Prove or disprove: for any integer x, one of the integers x, $x+5$, $x+7$ is divisible by 3.

(d) Formulate a (necessary and sufficient) condition on a, b such that the following statement is true: for any integer x, one of the integers x, $x+a$, $x+b$ is divisible by 3.

Solution. Since we are dividing by 3, it is natural to always distinguish the three cases $x=3q$, $x=3q+1$ and $x=3q+2$.

(a) This is true:

- If $x=3q$, then x itself is divisible by 3.
- If $x=3q+1$, then $x+2=3q+3$ is divisible by 3.
- If $x=3q+2$, then $x+4=3q+6$ is divisible by 3.

In each case, one of the integers x, $x+2$, $x+4$ is divisible by 3.

(b) This is false: for instance, if $x=2$, then none of x, $x+2$, $x+8$ is divisible by 3.

(c) This is also true:

- If $x=3q$, then x itself is divisible by 3.
- If $x=3q+1$, then $x+5=3q+6$ is divisible by 3.
- If $x=3q+2$, then $x+7=3q+9$ is divisible by 3.

In each case, one of the integers x, $x+5$, $x+7$ is divisible by 3.

(d) The thing that's different about the second case is the following: $x+2$ and $x+8=(x+2)+6$ leave the same remainder when dividing by 3.

With that in mind, we arrive at the following condition: for any integer x, one of the integers x, $x+a$, $x+b$ is divisible by 3 if and only if the remainders of a and b when dividing by 3 are such that one of them is 1 and the other is 2.

[Once formulated, this statement is proved in exactly the same way as the first and third case.] $\square$

Problem 5. Let $n \geqslant 0$ be an integer. Using induction, prove the following divisibility statements:

(a) $8|5^{2n}+7$ *Hint:* $5^{2(n+1)}+7=5^2(5^{2n}+7)+(7-5^2\cdot 7)$

(b) $15|2^{4n}-1$

Solution.

(a) We use induction on the claim that $8|(5^{2n}+7)$.

- The base case ($n=0$) is that $8|(1+7)$. That's true.
- For the inductive step, assume that $8|(5^{2n}+7)$ is true for some value of n.

  We need to show that $8|(5^{2(n+1)+2}+7)$ as well. Indeed,

  $$\begin{aligned} 5^{2(n+1)}+7 &= 25\cdot 5^{2n}+7 \\ &= 25(5^{2n}+7)-25\cdot 7+7 \\ &= 25(5^{2n}+7)-24\cdot 7 \end{aligned}$$

  is divisible by 8 because:

  - $8|24\cdot 7$ for obvious reasons, and
  - $8|25(5^{2n}+7)$ by the induction hypothesis.

Armin Straub
straub@southalabama.edu

By induction, it follows that $8|(5^{2n}+7)$ for all integers $n \geqslant 0$.

(b) We use induction on the claim that $15|(2^{4n}-1)$.

- The base case ($n=0$) is that $15|(1-1)$. That's true.
- For the inductive step, assume that $15|(2^{4n}-1)$ is true for some value of n.

  We need to show that $15|(2^{4(n+1)}-1)$ as well. Indeed,

$$\begin{aligned} 2^{4(n+1)}-1 &= 2^4 \cdot 2^{4n}-1 \\ &= 2^4(2^{4n}-1)+2^4-1 \\ &= 2^4(2^{4n}-1)+15 \end{aligned}$$

  is divisible by 15 because:

  - $15|15$ for obvious reasons, and
  - $15|2^4(2^{4n}-1)$ by the induction hypothesis.

By induction, it follows that $15|(2^{4n}-1)$ for all integers $n \geqslant 0$. □

Armin Straub
straub@southalabama.edu