Review. SVD

Example 165. Determine the SVD of $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$.

Comment. In contrast to our previous example, $\operatorname{rank}(A) = 1$. It follows that $A^T A$ has eigenvalue 0, so that 0 is a singular value of A.

Solution. $A^T A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$ has 10-eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and 0-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

We conclude that $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$.

$\boldsymbol{u}_1 = \frac{1}{\sigma_1} A \boldsymbol{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{20}} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

We cannot obtain $\boldsymbol{u}_2$ in the same way because $\sigma_2 = 0$. Since for every vector $\boldsymbol{u}_2$, $A\boldsymbol{v}_2 = \sigma_2 \boldsymbol{u}_2$, we can choose $\boldsymbol{u}_2$ as we wish, as long as the columns of U are orthonormal in the end.

$\boldsymbol{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ (but $\boldsymbol{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ works just as well)

Hence, $U = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

In summary, $A = U \Sigma V^T$ with $U = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Check. Do check that, indeed, $A = U \Sigma V^T$.

Example 166. Determine the SVD of $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution. $A^T A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ has 3-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and 1-eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Since $A^T A = V \Sigma^T \Sigma V^T$, we conclude that $V = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

$\boldsymbol{u}_1 = \frac{1}{\sigma_1} A \boldsymbol{v}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$

$\boldsymbol{u}_2 = \frac{1}{\sigma_2} A \boldsymbol{v}_2 = \frac{1}{1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\boldsymbol{u}_3$ is chosen so that the matrix U is orthogonal. Hence, $\boldsymbol{u}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ (or $\boldsymbol{u}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$).

Hence, $U = \begin{bmatrix} -2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$.

In summary, $A = U \Sigma V^T$ with $U = \begin{bmatrix} -2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$.

How did we find $\boldsymbol{u}_3$? We already have the vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, and need a vector orthogonal to both.

That is, we need to find the vector spanning $\operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}^{\perp} = \operatorname{col}\left(\begin{bmatrix} -2 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} \right)^{\perp} = \operatorname{null}\left(\begin{bmatrix} -2 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \right)$.

[Without the intermediate steps, can you see why the null space consists of precisely the vectors orthogonal to both $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$?]

More generally, proceeding like this, we can always fill in "missing" vectors $\boldsymbol{u}_i$ to obtain an orthonormal basis $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ that we can use as the columns of U.

Armin Straub
straub@southalabama.edu