Homework Set 9

Problem 1

Example 8. How many different Jordan normal forms are there for a 6×6 matrix with eigenvalues $4,4,4,4,9,9$?

Solution. One eigenvalue has multiplicity 4, the other multiplicity 2.

Therefore, there are $5\cdot 2=10$ possible Jordan normal forms.

[See Example 135 in Lecture 25 for where the 5 and 2 come from. Alternatively, you can see this from the listing that follows below.]

Listing all of them.

$$\begin{bmatrix} 4 &&&&& \\ & 4 &&&& \\ && 4 &&& \\ &&& 4 && \\ &&&& 9 & \\ &&&&& 9 \end{bmatrix}, \begin{bmatrix} 4 & 1 &&&& \\ & 4 &&&& \\ && 4 &&& \\ &&& 4 && \\ &&&& 9 & \\ &&&&& 9 \end{bmatrix}, \begin{bmatrix} 4 & 1 &&&& \\ & 4 &&&& \\ && 4 & 1 && \\ &&& 4 && \\ &&&& 9 & \\ &&&&& 9 \end{bmatrix}, \begin{bmatrix} 4 & 1 &&&& \\ & 4 & 1 &&& \\ && 4 &&& \\ &&& 4 && \\ &&&& 9 & \\ &&&&& 9 \end{bmatrix}, \begin{bmatrix} 4 & 1 &&&& \\ & 4 & 1 &&& \\ && 4 & 1 && \\ &&& 4 && \\ &&&& 9 & \\ &&&&& 9 \end{bmatrix},$$

$$\begin{bmatrix} 4 &&&&& \\ & 4 &&&& \\ && 4 &&& \\ &&& 4 && \\ &&&& 9 & 1 \\ &&&&& 9 \end{bmatrix}, \begin{bmatrix} 4 & 1 &&&& \\ & 4 &&&& \\ && 4 &&& \\ &&& 4 && \\ &&&& 9 & 1 \\ &&&&& 9 \end{bmatrix}, \begin{bmatrix} 4 & 1 &&&& \\ & 4 &&&& \\ && 4 & 1 && \\ &&& 4 && \\ &&&& 9 & 1 \\ &&&&& 9 \end{bmatrix}, \begin{bmatrix} 4 & 1 &&&& \\ & 4 & 1 &&& \\ && 4 &&& \\ &&& 4 && \\ &&&& 9 & 1 \\ &&&&& 9 \end{bmatrix}, \begin{bmatrix} 4 & 1 &&&& \\ & 4 & 1 &&& \\ && 4 & 1 && \\ &&& 4 && \\ &&&& 9 & 1 \\ &&&&& 9 \end{bmatrix}$$

Problem 2

Example 9. How many different Jordan normal forms are there for a 14×14 matrix with eigenvalues $3,4,4,4,4,5,5,5,5,7,7,7,7,8$?

Solution. The multiplicities of the eigenvalues are $1,4,4,4,1$.

Hence, there are $1\cdot 5\cdot 5\cdot 5\cdot 1=125$ possible Jordan normal forms.

Problem 3

Example 10. Solve the initial value problem $\boldsymbol{y}'=\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}\boldsymbol{y}$ with $\boldsymbol{y}(0)=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Solution. We can see right away that $A=\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ is not diagonalizable (because it is a 2×2 Jordan block).

The solution to the differential equation is

$$\begin{aligned}
\boldsymbol{y}(t) &= e^{At}\begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= e^{5It+Nt}\begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \text{with } N=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= e^{5It}e^{Nt}\begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \text{(because } 5It \text{ and } Nt \text{ commute)} \\
&= \begin{bmatrix} e^{5t} & \\ & e^{5t} \end{bmatrix}\left(1+Nt+\frac{1}{2}(Nt)^2+\frac{1}{3!}(Nt)^3+\ldots\right)\begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} e^{5t} & \\ & e^{5t} \end{bmatrix}(1+Nt)\begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \text{(because } N^2=\mathbf{0}\text{)} \\
&= \begin{bmatrix} e^{5t} & \\ & e^{5t} \end{bmatrix}\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} e^{5t} & \\ & e^{5t} \end{bmatrix}\begin{bmatrix} 1-2t \\ -2 \end{bmatrix}=\begin{bmatrix} (1-2t)e^{5t} \\ -2e^{5t} \end{bmatrix}.
\end{aligned}$$

Problem 4

Example 11. Determine the 2×2 matrix Q for rotation by 49 degrees.

Solution. Recall that the rotation matrix for angle θ is $Q=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ (see Example 145 in Lecture 27).

Hence, $Q=\begin{bmatrix}\cos(49^\circ) & -\sin(49^\circ)\\ \sin(49^\circ) & \cos(49^\circ)\end{bmatrix}=\begin{bmatrix}\cos\left(49\cdot\frac{2\pi}{360}\right) & -\sin\left(49\cdot\frac{2\pi}{360}\right)\\ \sin\left(49\cdot\frac{2\pi}{360}\right) & \cos\left(49\cdot\frac{2\pi}{360}\right)\end{bmatrix}\approx\begin{bmatrix}0.6561 & -0.7547\\ 0.7547 & 0.6561\end{bmatrix}$.

Problem 5

Example 12. Determine the square of the norm of the vector $\begin{bmatrix}5-5i\\ 1+3i\end{bmatrix}$.

Solution. $\left\|\begin{bmatrix}5-5i\\ 1+3i\end{bmatrix}\right\|^2=|5-5i|^2+|1+3i|^2=(5^2+(-5)^2)+(1^2+3^2)=50+10=60$

Problem 6

Example 13. Given $A=\begin{bmatrix}2-i & -1+i\\ 3-3i & 2+3i\end{bmatrix}$, determine A^*.

Solution. $A^*=(\bar{A})^T=\begin{bmatrix}2+i & -1-i\\ 3+3i & 2-3i\end{bmatrix}^T=\begin{bmatrix}2+i & 3+3i\\ -1-i & 2-3i\end{bmatrix}$

Problem 7

Example 14. Invert the complex number $5-4i$.

Solution. $\frac{1}{5-4i}=\frac{5+4i}{(5-4i)(5+4i)}=\frac{5+4i}{5^2-(4i)^2}=\frac{5+4i}{25+16}=\frac{5}{41}+\frac{4}{41}i$

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