Review. SVD

Example 145. Determine the SVD of $A=\begin{bmatrix} 2 & 2\\ 1 & 1 \end{bmatrix}$.

Comment. In contrast to our previous example, $\mathrm{rank}(A)=1$. It follows that A^TA has eigenvalue 0, so that 0 is a singular value of A.

Solution. $A^TA=\begin{bmatrix} 5 & 5\\ 5 & 5 \end{bmatrix}$ has 10-eigenvector $\begin{bmatrix} 1\\ 1 \end{bmatrix}$ and 0-eigenvector $\begin{bmatrix} -1\\ 1 \end{bmatrix}$.

We conclude that $V=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1\\ 1 & 1 \end{bmatrix}$ and $\Sigma=\begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$.

$\boldsymbol{u}_1=\frac{1}{\sigma_1}A\boldsymbol{v}_1=\frac{1}{\sqrt{10}}\begin{bmatrix} 2 & 2\\ 1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1\\ 1 \end{bmatrix}=\frac{1}{\sqrt{20}}\begin{bmatrix} 4\\ 2 \end{bmatrix}=\frac{1}{\sqrt{5}}\begin{bmatrix} 2\\ 1 \end{bmatrix}$

We cannot obtain $\boldsymbol{u}_2$ in the same way because $\sigma_2=0$. Since for every vector $\boldsymbol{u}_2$, $A\boldsymbol{v}_2=\sigma_2\boldsymbol{u}_2$, we can choose $\boldsymbol{u}_2$ as we wish, as long as the columns of U are orthonormal in the end.

$\boldsymbol{u}_2=\frac{1}{\sqrt{5}}\begin{bmatrix} -1\\ 2 \end{bmatrix}$ (but $\boldsymbol{u}_2=\frac{1}{\sqrt{5}}\begin{bmatrix} 1\\ -2 \end{bmatrix}$ works just as well)

Hence, $U=\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1\\ 1 & 2 \end{bmatrix}$.

In summary, $A=U\Sigma V^T$ with $U=\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1\\ 1 & 2 \end{bmatrix}$, $\Sigma=\begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$, $V=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1\\ 1 & 1 \end{bmatrix}$.

Check. Do check that, indeed, $A=U\Sigma V^T$.

Example 146. Determine the SVD of $A=\begin{bmatrix} 1 & -1\\ 0 & 1\\ 1 & 0 \end{bmatrix}$.

Solution. $A^TA=\begin{bmatrix} 2 & -1\\ -1 & 2 \end{bmatrix}$ has 3-eigenvector $\begin{bmatrix} -1\\ 1 \end{bmatrix}$ and 1-eigenvector $\begin{bmatrix} 1\\ 1 \end{bmatrix}$.

Since $A^TA=V\Sigma^T\Sigma V^T$, we conclude that $V=\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1\\ 1 & 1 \end{bmatrix}$ and $\Sigma=\begin{bmatrix} \sqrt{3} & 0\\ 0 & 1\\ 0 & 0 \end{bmatrix}$.

$\boldsymbol{u}_1=\frac{1}{\sigma_1}A\boldsymbol{v}_1=\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1\\ 0 & 1\\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1\\ 1 \end{bmatrix}=\frac{1}{\sqrt{6}}\begin{bmatrix} -2\\ 1\\ -1 \end{bmatrix}$

$\boldsymbol{u}_2=\frac{1}{\sigma_2}A\boldsymbol{v}_2=\frac{1}{1}\begin{bmatrix} 1 & -1\\ 0 & 1\\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1\\ 1 \end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix} 0\\ 1\\ 1 \end{bmatrix}$

$\boldsymbol{u}_3$ is chosen so that the matrix U is orthogonal. Hence, $\boldsymbol{u}_3=\frac{1}{\sqrt{3}}\begin{bmatrix} -1\\ -1\\ 1 \end{bmatrix}$ (or $\boldsymbol{u}_3=\frac{1}{\sqrt{3}}\begin{bmatrix} 1\\ 1\\ -1 \end{bmatrix}$).

Hence, $U=\begin{bmatrix} -2/\sqrt{6} & 0 & -1/\sqrt{3}\\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3}\\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$.

In summary, $A=U\Sigma V^T$ with $U=\begin{bmatrix} -2/\sqrt{6} & 0 & -1/\sqrt{3}\\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3}\\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$, $\Sigma=\begin{bmatrix} \sqrt{3} & 0\\ 0 & 1\\ 0 & 0 \end{bmatrix}$, $V=\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1\\ 1 & 1 \end{bmatrix}$.

How did we find $\boldsymbol{u}_3$? We already have the vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, and need a vector orthogonal to both.

That is, we need to find the vector spanning $\mathrm{span}\left\{\begin{bmatrix} -2\\ 1\\ -1 \end{bmatrix},\begin{bmatrix} 0\\ 1\\ 1 \end{bmatrix}\right\}^{\perp}=\mathrm{col}\left(\begin{bmatrix} -2 & 0\\ 1 & 1\\ -1 & 1 \end{bmatrix}\right)^{\perp}=\mathrm{null}\left(\begin{bmatrix} -2 & 1 & -1\\ 0 & 1 & 1 \end{bmatrix}\right)$.

[Without the intermediate steps, can you see why the null space consists of precisely the vectors orthogonal to both $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$?]

More generally, proceeding like this, we can always fill in "missing" vectors $\boldsymbol{u}_i$ to obtain an orthonormal basis $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_m$ that we can use as the columns of U.

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