Review. Fibonacci numbers, Binet formula

Definition 95. A sequence a_n satisfying a recursion of the form

$$a_{n+d} = c_1 a_{n+d-1} + c_2 a_{n+d-2} + \ldots + c_d a_n$$

is called **C-finite** of order d.

For instance. For the Fibonacci numbers, $d = 2$ and $c_1 = c_2 = 1$.

In matrix-vector form. $\begin{bmatrix} a_{n+d} \\ a_{n+d-1} \\ \vdots \\ a_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} c_1 & c_2 & \cdots & c_{d-1} & c_d \\ 1 & & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \end{bmatrix}}_{T} \begin{bmatrix} a_{n+d-1} \\ a_{n+d-2} \\ \vdots \\ a_n \end{bmatrix}$

By the same reasoning as for Fibonacci numbers, C-finite sequences have a Binet-like formula:

Theorem 96. (generalized Binet formula) Suppose the recursion matrix T has distinct eigenvalues $\lambda_1, \ldots, \lambda_d$. Then

$$a_n = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \ldots + \alpha_d \lambda_d^n$$

for some fixed numbers $\alpha_1, \ldots, \alpha_d$.

For instance. For the Fibonacci numbers, $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$, and $\alpha_1 = \frac{1}{\sqrt{5}}$, $\alpha_2 = -\frac{1}{\sqrt{5}}$.

Comment. A little more care is needed in the case that eigenvalues are repeated.

Corollary 97. Under the assumptions of the previous theorem, if λ_1 is the eigenvalue with the largest absolute value and $\lambda_1 > 0$, as well as $\alpha_1 \neq 0$, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda_1$.

Proof. This follows from $a_n = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \ldots + \alpha_d \lambda_d^n$ because, for large n, the term $\alpha_1 \lambda_1$ dominates the others. Indeed, we have

$$\frac{a_{n+1}}{a_n} = \frac{\alpha_1 \lambda_1^{n+1} + \alpha_2 \lambda_2^{n+1} + \ldots + \alpha_d \lambda_d^{n+1}}{\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \ldots + \alpha_d \lambda_d^n} = \frac{\alpha_1 \lambda_1 + \alpha_2 \lambda_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n + \ldots + \alpha_d \lambda_d \left(\frac{\lambda_d}{\lambda_1}\right)^n}{\alpha_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n + \ldots + \alpha_d \left(\frac{\lambda_d}{\lambda_1}\right)^n} \overset{n\to\infty}{\longrightarrow} \frac{\alpha_1 \lambda_1}{\alpha_1} = \lambda_1.$$

□

Example 98. Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 5a_n$ and $a_0 = 0$, $a_1 = 1$.

(a) Determine the first few terms of the sequence.

(b) Find a Binet-like formula for a_n.

(c) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

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Solution.

(a) $0, 1, 2, 9, 28, 101, 342, 1189, 4088, \ldots$

(b) The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

The eigenvalues of $\begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix}$ are $1 \pm \sqrt{6}$.

Hence, $a_n = \alpha_1 (1 + \sqrt{6})^n + \alpha_2 (1 - \sqrt{6})^n$ and we only need to figure out the two unknowns α_1, α_2. We can do that using the two initial conditions: $a_0 = \alpha_1 + \alpha_2 = 0$, $a_1 = \alpha_1 (1 + \sqrt{6}) + \alpha_2 (1 - \sqrt{6}) = 1$.

Solving, we find $\alpha_1 = \frac{1}{2\sqrt{6}}$ and $\alpha_2 = -\frac{1}{2\sqrt{6}}$ so that, in conclusion, $a_n = \frac{(1+\sqrt{6})^n - (1-\sqrt{6})^n}{2\sqrt{6}}$.

Comment. Alternatively, we could have proceeded as we did last time in the case of the Fibonacci numbers: starting with the recursion matrix T, we compute its diagonalization $T = PDP^{-1}$. Multiplying out $PD^n P^{-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$, we obtain the Binet-like formula for a_n. However, this is more work than what we did.

(c) It follows from the Binet-like formula that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{6} \approx 3.44949$.

Comment. Actually, we don't even need the Binet-like formula. Just the eigenvalues and the observation that α_1 cannot be 0 are enough. [We cannot have $\alpha_1 = 0$, because then $a_n = \alpha_2 (1 - \sqrt{6})^n$ so that $a_0 = 0$ would imply $\alpha_2 = 0$.]

Example 99. (homework) Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 4a_n$ and $a_0 = 0$, $a_1 = 1$. Determine $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$.

First few terms of sequence. $0, 1, 2, 8, 24, 80, 256, 832, \ldots$

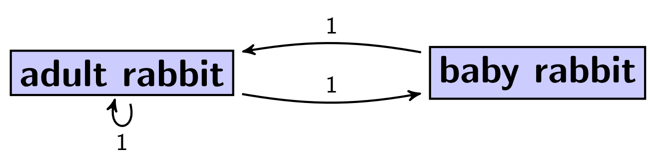
These are actually related to Fibonacci numbers. Indeed, $a_n = 2^{n-1} F_n$. Can you prove this directly from the recursions? Alternatively, this follows from the Binet formulas.

Solution. Proceeding as in the previous example, we find $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{5} \approx 3.23607$.

Comment. With just a little more work, we find the Binet formula $a_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2\sqrt{5}}$.

Example 100. We model rabbit reproduction as follows.

Each month, every pair of adult rabbits produces one pair of baby rabbit as offspring. Meanwhile, it takes baby rabbits one month to mature to adults.



Comment. In this simplified model, rabbits always come in male/female pairs and no rabbits die. Though these feautures might make it sound fairly useless, the model may have some merit when describing populations under ideal conditions (unlimited resources) and over short time (no deaths).

Historical comment. The question how many rabbits there are after one year, when starting out with a pair of baby rabbits is famously included in the 1202 textbook of the Italian mathematician Leonardo of Pisa, known as Fibonacci.

Describe the transition from one month to the next.

Solution. Let x_t be the number of adult rabbit pairs after t months. Likewise, y_t is the number of baby rabbit pairs. Then the transition from one month to the next is described by

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} x_t + y_t \\ x_y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_y \end{bmatrix}.$$

That's precisely the transition for the Fibonacci numbers!

It follows that Fibonacci numbers count the number of rabbits in this model.

Comment. Note that the setup is very much as for Markov chains. Here, however, the outgoing values do not add to 100% for each state. Consequently, we cannot expect an equilibrium (and, indeed, the number of rabbits increases without bound).

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