Example 10. (review) If $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, then its **transpose** is $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Recall that $(AB)^T = B^T A^T$. This reflects the fact that, in the column-centric versus the row-centric interpretation of matrix multiplication, the order of the matrices is reversed.

Comment. When working with complex numbers, the fundamental role is not played by the transpose but by the **conjugate transpose** instead (we'll see that in our discussion of orthogonality): $A^* = \overline{A^T}$.

For instance, if $A = \begin{bmatrix} 1-3i & 5i \\ 2+i & 3 \end{bmatrix}$, then $A^* = \begin{bmatrix} 1+3i & 2-i \\ -5i & 3 \end{bmatrix}$.

Example 11. (review) $\mathbb{R}^3$ is the vector space of all vectors with 3 real entries.

$\mathbb{R}$ itself refers to the set of real numbers. We will later also discuss $\mathbb{C}$, the set of complex numbers.

The **standard basis** of $\mathbb{R}^3$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Recall that vectors from a vector space V form a basis of V if and only iff

- the vectors span V, and
- the vectors are (linearly) independent.

Make sure that you can define precisely what it means for vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ to be independent.

Suppose that A is $n \times n$ and has independent eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$.
Then A can be **diagonalized** as $A = PDP^{-1}$, where

- the columns of P are the eigenvectors, and
- the diagonal matrix D has the eigenvalues on the diagonal

Such a diagonalization is possible if and only if A has enough (independent) eigenvectors.

Comment. If you don't quite recall why these choices result in the diagonalization $A = PDP^{-1}$, note that the diagonalization is equivalent to $AP = PD$.

- Put the eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ as columns into a matrix P.

$$A\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i \quad \Longrightarrow \quad A \begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1 \boldsymbol{x}_1 & \cdots & \lambda_n \boldsymbol{x}_n \\ | & & | \end{bmatrix}$$
$$= \begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

- In summary: $AP = PD$

Example 12. (extra practice) Diagonalize, if possible, the matrices

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 2 & 0 \\ 1 & 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution. For instance, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & -4 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & & \\ & 2 & \\ & & 2 \end{bmatrix}$. B is not diagonalizable.

For instance, $C = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$.

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Example 13. (to be finished next time) Let $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$.

(a) Find the eigenvalues and bases for the eigenspaces of A.

(b) Diagonalize A. That is, determine matrices P and D such that $A = PDP^{-1}$.

Solution.

(a) By expanding by the second column, we find that the characteristic polynomial $\det(A - \lambda I)$ is

$$\begin{vmatrix} 4-\lambda & 0 & 2 \\ 2 & 2-\lambda & 2 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)[(4-\lambda)(3-\lambda)-2] = (2-\lambda)^2(5-\lambda).$$

Hence, the eigenvalues are $\lambda = 2$ (with multiplicity 2) and $\lambda = 5$.

Comment. At this point, we know that we will find one eigenvector for $\lambda = 5$ (more precisely, the 5-eigenspace definitely has dimension 1). On the other hand, the 2-eigenspace might have dimension 2 or 1. In order for A to be diagonalizable, the 2-eigenspace must have dimension 2. (Why?!)

- The 5-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}\right)$.

  Doing one set of row operations, we obtain

  $$\operatorname{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}\right) \overset{\substack{R_2+2R_1 \Rightarrow R_2 \\ R_3+R_1 \Rightarrow R_3}}{=} \operatorname{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right\}.$$

  In other words, the 5-eigenspace has basis $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

  Review. The **row-reduced echelon form (RREF)** of $\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$.

- The 2-eigenspace is $\operatorname{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right)$.

  $$\operatorname{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right) \overset{\substack{R_2-R_1 \Rightarrow R_2 \\ R_3-\frac{1}{2}R_1 \Rightarrow R_3}}{=} \operatorname{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

  In other words, the 2-eigenspace has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

  Comment. So, indeed, the 2-eigenspace has dimension 2. In particular, A is diagonalizable.

  Review. By our computation, and scaling the first row, the RREF of $\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) A possible choice is $P = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Comment. However, many other choices are possible and correct. For instance, the order of the eigenvalues in D doesn't matter (as long as the same order is used for P). Also, for P, the columns can be chosen to be any other set of eigenvectors.