Example 129. (last example, cont'd) We will write $T=\begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$ for the transition matrix.

- If we start with proportions $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ of people employed/unemployed, what will be the proportions after 2 years? (n years?)

  Solution. After 1 year, $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}=T\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$. After 2 years, $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}=T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}=T^2\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

  After n years, $\begin{bmatrix} x_n \\ y_n \end{bmatrix}=T^n\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

- Taken together, the 1-eigenvector $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and the 0.4-eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ form a basis of $\mathbb{R}^2$.

  This means that we can express any vector (uniquely) as a combination of eigenvectors!

- Express the vector $\frac{1}{4}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in terms of the basis $\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

  Solution. In other words, we need to solve $\begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix}\boldsymbol{a}=\frac{1}{4}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

  The (unique) solution is $\boldsymbol{a}=\begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix}^{-1}\frac{1}{4}\begin{bmatrix} 3 \\ 1 \end{bmatrix}=\frac{1}{6}\begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}\frac{1}{4}\begin{bmatrix} 3 \\ 1 \end{bmatrix}=\frac{1}{24}\begin{bmatrix} 4 \\ 2 \end{bmatrix}=\begin{bmatrix} 1/6 \\ 1/12 \end{bmatrix}$.

  That is, $\frac{1}{4}\begin{bmatrix} 3 \\ 1 \end{bmatrix}=\frac{1}{6}\begin{bmatrix} 5 \\ 1 \end{bmatrix}+\frac{1}{12}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

  Important comment. This is a **change of basis**! When using the coordinates $3/4, 1/4$, we are (silently) working with respect to the standard basis: $\frac{1}{4}\begin{bmatrix} 3 \\ 1 \end{bmatrix}=\frac{3}{4}\begin{bmatrix} 1 \\ 0 \end{bmatrix}+\frac{1}{4}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

  We will see below that, for our present purposes, it is much more convenient to work with the coordinates $1/6, 1/12$ with respect to the basis of eigenvectors: $\frac{1}{4}\begin{bmatrix} 3 \\ 1 \end{bmatrix}=\frac{1}{6}\begin{bmatrix} 5 \\ 1 \end{bmatrix}+\frac{1}{12}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

- If $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}=\frac{1}{4}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, then what are $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_{10} \\ y_{10} \end{bmatrix}$?

  Comment. Since $\begin{bmatrix} x_n \\ y_n \end{bmatrix}=T^n\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, we could just repeatedly multiply with the matrix T.

  For instance, $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}=\begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}\frac{1}{4}\begin{bmatrix} 3 \\ 1 \end{bmatrix}=\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}=\begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}=\begin{bmatrix} 0.82 \\ 0.18 \end{bmatrix}$.

  However, this is not going to be practical for computing $\begin{bmatrix} x_{10} \\ y_{10} \end{bmatrix}$.

  Solution.

  $$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}=T\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}=T\left(\frac{1}{6}\begin{bmatrix} 5 \\ 1 \end{bmatrix}+\frac{1}{12}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)=1\cdot\frac{1}{6}\begin{bmatrix} 5 \\ 1 \end{bmatrix}+0.4\cdot\frac{1}{12}\begin{bmatrix} -1 \\ 1 \end{bmatrix}=\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

  $$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}=T^2\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}=T^2\left(\frac{1}{6}\begin{bmatrix} 5 \\ 1 \end{bmatrix}+\frac{1}{12}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)=1^2\cdot\frac{1}{6}\begin{bmatrix} 5 \\ 1 \end{bmatrix}+0.4^2\cdot\frac{1}{12}\begin{bmatrix} -1 \\ 1 \end{bmatrix}=\begin{bmatrix} 0.82 \\ 0.18 \end{bmatrix}$$

  $$\begin{bmatrix} x_{10} \\ y_{10} \end{bmatrix}=T^{10}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}=T^{10}\left(\frac{1}{6}\begin{bmatrix} 5 \\ 1 \end{bmatrix}+\frac{1}{12}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)=1^{10}\cdot\frac{1}{6}\begin{bmatrix} 5 \\ 1 \end{bmatrix}+0.4^{10}\cdot\frac{1}{12}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\approx\begin{bmatrix} 0.833 \\ 0.167 \end{bmatrix}$$

  This almost equals $\frac{1}{6}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ because $0.4^{10}\approx 0.000105$ is tiny.

  Important observation. You can now nicely observe how our initial state is quickly approaching the equilibrium state. Moreover, you can see that this will happen for **any** initial state!

Example 130. If $A\boldsymbol{x}=\lambda\boldsymbol{x}$, then what is $A^{100}\boldsymbol{x}$?

Solution. $A^{100}\boldsymbol{x}=\lambda^{100}\boldsymbol{x}$

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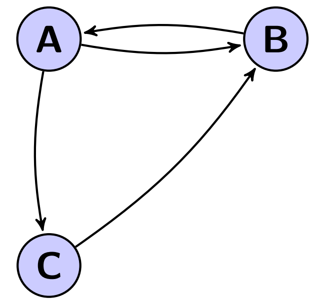
Example 131. Suppose the internet consists of only the three webpages A, B, C.

We wish to rank these webpages in order of "importance".



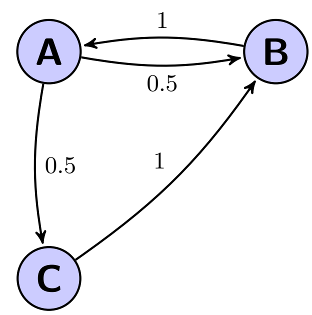
The idea. Instead of analyzing each webpage (which would be a lot of work!) we will try to only use the information how the pages are linked to each other. The idea being that an "important" page should be linked to from many other pages.

A and B have a link to each other. Also, A links to C and C links to B. If you keep randomly clicking from one webpage to the next, what proportion of the time will you be at each page?

The idea. We will assign ranking to those pages according to how frequently such a random surfer would visit these pages.

Comment. Before we start computing, stop for a moment, and think about how you would rank the webpages.

Solution. Let a_t be the probability that we will be on page A at time t. Likewise, b_t, c_t are the probabilities that we will be on page B or C.



The transition from one state to the next now works exactly as in the previous example. We get the following transition matrix:

$$\begin{bmatrix} a_{t+1} \\ b_{t+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \cdot a_t + 1 \cdot b_t + 0 \cdot c_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 1 \cdot c_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 0 \cdot c_t \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} a_t \\ b_t \\ c_t \end{bmatrix}$$

To find the equilibrium state, we again determine an appropriate 1-eigenvector.

The 1-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \\ \frac{1}{2} & 0 & -1 \end{bmatrix} \right)$ which has basis $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

The corresponding equilibrium state is $\frac{1}{5}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. In this context, this is also known as the **PageRank vector**.

In other words, after browsing randomly for a long time, there is (about) a $\frac{2}{5} = 40\%$ chance to be at page A, a $\frac{2}{5} = 40\%$ chance to be at page B, and a $\frac{1}{5} = 20\%$ chance to be at page C.

We therefore rank A and B highest (tied), and C lowest.

Just checking. Maybe we were expecting B to be ranked above A, because B is the only page that has two incoming links. However, if we are at page B, then our next click will be to page A, which is why A and B receive equal ranking.

This method of ranking is the famous **PageRank** algorithm (underlying Google's search algorithm).

By the way, the algorithm is named, not after ranking web"pages", but after Larry Page (who founded Google in 1998 together with Sergey Brin).

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