Example 127. Find the eigenvalues of $A = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & -1 \\ 1 & 0 & 3 \end{bmatrix}$ as well as bases for the eigenspaces.

Solution. By expanding by the second column, we find that the characteristic polynomial is

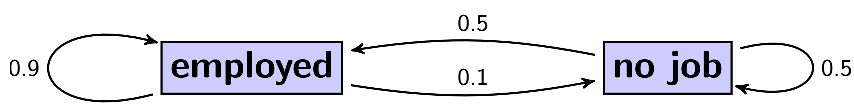
$$\begin{vmatrix} 3-\lambda & 0 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)[(3-\lambda)^2 - 1] = (2-\lambda)(\lambda-2)(\lambda-4).$$

Since $\lambda = 2$ is a double root, we say that it has **(algebraic) multiplicity** 2.
Hence, the eigenvalues are $\lambda = 2$ (with multiplicity 2) and $\lambda = 4$.

- For $\lambda = 4$, the eigenspace $\mathrm{null}\left(\begin{bmatrix} -1 & 0 & 1 \\ -1 & -2 & -1 \\ 1 & 0 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.
- For $\lambda = 2$, the eigenspace $\mathrm{null}\left(\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Example 128. Consider a fixed population of people with or without a job. Suppose that, each year, 50% of those unemployed find a job while 10% of those employed lose their job. What is the unemployment rate in the long term equilibrium?

Solution.



x_t: proportion of population employed at time t (in years)
y_t: proportion of population unemployed at time t [Note that $x_t + y_t = 1$.]

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0.9x_t + 0.5y_t \\ 0.1x_t + 0.5y_t \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}\begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

The matrix $\begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$ is the **transition matrix** of this dynamical system, because it describes the transition from time t to time $t+1$. This particular one is a **Markov matrix** (or stochastic matrix): its columns add to 1 and it has no negative entries.

$\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an equilibrium if $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$. In other words, $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an eigenvector with eigenvalue 1.

The 1-eigenspace is $\mathrm{null}\left(\begin{bmatrix} -0.1 & 0.5 \\ 0.1 & -0.5 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

Since $x_\infty + y_\infty = 1$, we conclude that $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \frac{1}{5+1}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/6 \\ 1/6 \end{bmatrix}$.

Hence, the unemployment rate in the long term equilibrium is $1/6 \approx 16.7\%$.
[Ponder about why this is a reasonable value!]

Comment. What's the other eigenvalue of the transition matrix? No need to compute the characteristic polynomial: we can easily see that it is 0.4 because the product of the eigenvalues equals the determinant!
The 0.4-eigenspace is spanned by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Advanced comment. Will the employment and unemployment rate always stabilize (and thus approach the long term equilibrium)? Yes! This is a consequence of the other eigenvalue of the transition matrix satisfying $|0.4| < 1$. If we start in state $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = a\begin{bmatrix} 5 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = 1^n \cdot a\begin{bmatrix} 5 \\ 1 \end{bmatrix} + (0.4)^n \cdot b\begin{bmatrix} -1 \\ 1 \end{bmatrix} \underset{\text{as } n\to\infty}{\longrightarrow} a\begin{bmatrix} 5 \\ 1 \end{bmatrix}$.