Theorem 57 tells us that, if A is invertible, then the linear system $A\boldsymbol{x}=\boldsymbol{b}$ has a unique solution for any choice of $\boldsymbol{b}$ (namely, $\boldsymbol{x}=A^{-1}\boldsymbol{b}$). This property characterizes invertible matrices!

Theorem 63.

The linear system $A\boldsymbol{x}=\boldsymbol{b}$ has a unique solution for any choice of $\boldsymbol{b}$.

$\iff$ The RREF of A is an identity matrix. (In particular, A has to be a square matrix!)

$\iff$ A is invertible.

Why? If $A\boldsymbol{x}=\boldsymbol{b}$ has a unique solution for any choice of $\boldsymbol{b}$, then the RREF of A cannot have a free variable, which means that every column of the RREF has to contain a pivot. Also, the RREF of A cannot have a zero row (because then the system will be inconsistent for certain $\boldsymbol{b}$), which means that every row of the RREF has to contain a pivot. But that means that the number of columns and rows has to be the same (both equal to the number of pivots) and, moreover that the RREF is all zeroes except 1's on the diagonal (the pivots). In other words, the RREF is an identity matrix.

On the other hand, A is invertible if and only if $[\ A \mid I\ ]$ has RREF $[\ I \mid A^{-1}\ ]$, which is the case precisely if the RREF of A by itself is I.

8 Determinants

As for inverses, only square matrices have determinants!

We will write both $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ and $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ for the determinant.

Goal: A is invertible $\iff$ $\det(A)\neq 0$

Example 64. The **determinant** of

- a 2×2 matrix is $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)=ad-bc$, recall that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- a 1×1 matrix is $\det([\ a\ ])=a$.

Definition 65. In general, the **determinant** is characterized by:

- the normalization $\det I=1$,
- and how it is affected by elementary row operations:
  - **(add)** $R_i+\lambda R_j \Rightarrow R_i$ does not change the determinant.
  - **(swap)** $R_j \Leftrightarrow R_i$ reverses the sign of the determinant.
  - **(scale)** $\lambda R_i \Rightarrow R_i$ multiplies the determinant by λ.

Example 66. Compute $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{vmatrix}$.

Solution.

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{vmatrix} \overset{\frac{1}{2}R_2\Rightarrow R_2}{=} 2\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 7 \end{vmatrix} \overset{\frac{1}{7}R_3\Rightarrow R_3}{=} 14\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \overset{\substack{R_1-3R_3\Rightarrow R_1 \\ R_2-2R_3\Rightarrow R_2}}{=} 14\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \overset{R_1-2R_2\Rightarrow R_1}{=} 14\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 14$$

The determinant of a triangular matrix is the product of the diagonal entries.

Example 67. $\begin{vmatrix} 3 & 2 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 7 \end{vmatrix} = 3 \cdot 4 \cdot 7 = 84$

Example 68. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 7 & 13 \end{vmatrix} = 0$

Why? Because we see that the first two rows are multiples of each other. Hence, the matrix is not invertible (after one step of Gaussian elimination, we already have a zero row), and so has determinant 0.

Example 69. Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$.

Solution.

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} \overset{\substack{R_2 - 3R_1 \Rightarrow R_2 \\ R_3 - 2R_1 \Rightarrow R_3}}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & -4 & 1 \end{vmatrix} \overset{R_3 - \frac{4}{7}R_2 \Rightarrow R_3}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & -\frac{1}{7} \end{vmatrix} = 1 \cdot (-7) \cdot \left(-\frac{1}{7}\right) = 1$$

8.1 A "bad" way to compute determinants

Example 70. Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution. We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} + & & \\ & -1 & 2 \\ & 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix}$$

$$\overset{\text{i.e.}}{=} 1 \cdot \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = 1 \cdot (-1) - 2 \cdot (-1) + 0 = 1$$

Each term in the cofactor expansion is ± 1 times an entry times a smaller determinant (row and column of entry deleted).

The ± 1 is assigned to each entry according to $\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$.

Solution. We expand by the second column:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -2 \cdot \begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 1 & & 0 \\ & + & \\ 2 & & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & & 0 \\ 3 & & 2 \\ & - & \end{vmatrix}$$

$$= -2 \cdot (-1) + (-1) \cdot 1 - 0 = 1$$

Remark 71. Why is the method of cofactor expansion not practical?

Because to compute a large $n \times n$ determinant,

- one reduces to n determinants of size $(n-1) \times (n-1)$,
- then $n(n-1)$ determinants of size $(n-2) \times (n-2)$,
- and so on.

In the end, we have $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ many numbers to add.

WAY TOO MUCH WORK! Already $25! = 15511210043330985984000000 \approx 1.55 \cdot 10^{25}$.

Context. Today's fastest computer (as of June 2016), Sunway TaihuLight, which is rated 93 petaflops (that's $9.3 \cdot 10^{16}$ operations per second), would need more than 5 years at full speed for computing that one determinant. Which, with a lot of patient, you can actually do by hand (if using Gaussian elimination).

Big: about 10 million cores. Costly: about 270 million USD. Uses: oil exploration, weather, drug, design

By the way: "fastest" is measured by doing Gaussian elimination! ("LINPACK benchmarks")

kilo: 10^3, mega: 10^6, giga: 10^9, tera: 10^{12}, peta: 10^{15}, exa: 10^{18} (followed by the more recent zetta, yotta)

Armin Straub
straub@southalabama.edu