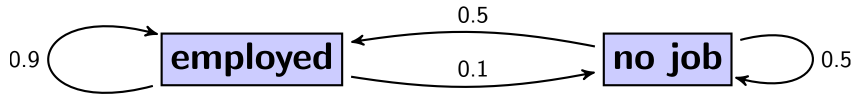
Sketch of Lecture 26 Tue, 12/01/2015

Review 158. Go over quiz problem, eigenvalues, eigenvectors.

Example 159. Consider a fixed population of people with or without a job. Suppose that, each year, 50% of those unemployed find a job while 10% of those employed loose their job. What is the unemployment rate in the long term equilibrium?

Solution.



x_t: proportion of population employed at time t (in years)

y_t: proportion of population unemployed at time t

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0.9x_t + 0.5y_t \\ 0.1x_t + 0.5y_t \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

The matrix $\begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$ is called a **Markov matrix** (or stochastic matrix): its columns add to 1 and it has no negative entries.

$\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an equilibrium if $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$.

In other words, $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an eigenvector with eigenvalue 1.

Eigenspace of $\lambda = 1$: $\operatorname{null}\left(\begin{bmatrix} -0.1 & 0.5 \\ 0.1 & -0.5 \end{bmatrix} \right) = \operatorname{span}\left\{ \begin{bmatrix} 5 \\ 1 \end{bmatrix} \right\}$

Since $x_\infty + y_\infty = 1$, we conclude that $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \frac{1}{5+1} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/6 \\ 1/6 \end{bmatrix}$.

Hence, the unemployment rate in the long term equilibrium is $1/6$.

Example 160. If $A\boldsymbol{x} = \lambda \boldsymbol{x}$, then $A^{100}\boldsymbol{x} = \lambda^{100}\boldsymbol{x}$.

Example 161. (just for fun) The famous **PageRank** algorithm (underlying Google's search algorithm) is based on the same kind of ideas about equilibrium: imagine a surfer randomly following links websites. At each time, the surfer is on one website (so there is a state for each website), and we can introduce a transition matrix for getting to the next website (the surfer clicks on one of the links in the current website). We can then determine the equilibrium vector as in our example above. This vector assigns a probability to each website: the higher the probability, the more important a website is deemed. That's the basic idea for ranking websites!

Armin Straub
straub@southalabama.edu

Example 162. (just for fun) Fibonacci numbers: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

Did you notice: $\frac{13}{8} = 1.625$, $\frac{21}{13} = 1.615$, $\frac{34}{21} = 1.619$, ...

The **golden ratio** $\varphi = 1.618\ldots$ Where's that coming from?

- $F_{n+1} = F_n + F_{n-1} \implies \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$
- $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is an eigenvalue of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

What's going on? Here is some more details (with more to fill in):

- $F_{n+1} = F_n + F_{n-1} \implies \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$
- Since $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$, we have $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$. $\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- The characteristic polynomial of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is $\lambda^2 - \lambda - 1$.
- The eigenvalues are $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$ (the golden ratio!) and $\lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618$.
- Corresponding eigenvectors: $\boldsymbol{v}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$
- Write $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \boldsymbol{v}_1 + c_1 \boldsymbol{v}_2$. $\left(c_1 = \frac{1}{\sqrt{5}}, c_2 = -\frac{1}{\sqrt{5}}\right)$
- $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1^n c_1 \boldsymbol{v}_1 + \lambda_2^n c_2 \boldsymbol{v}_2$
- Hence, $F_n = \lambda_1^n c_1 + \lambda_2^n c_2 = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$.

  That's **Binet's formula**.
- But $|\lambda_2| < 1$, and so $F_n \approx \lambda_1^n c_1 = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n$.

  In fact, $F_n = \mathrm{round}\left(\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n \right)$. Don't you feel powerful!?