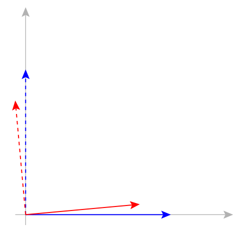
**Example 145.**

The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

... gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ x \end{bmatrix}$, i.e.

... rotates every vector in $\mathbb{R}^2$ counter-clockwise by $90°$.

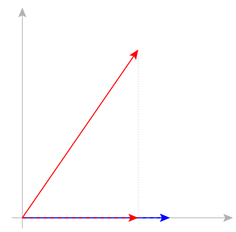


**Example 146.**

The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

... gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ 0 \end{bmatrix}$, i.e.

... projects every vector in $\mathbb{R}^2$ through onto the $x$-axis.



## Eigenvectors and eigenvalues

Throughout, $A$ will be an $n \times n$ matrix.

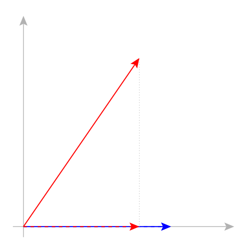
**Definition 147.** If $A\boldsymbol{x} = \lambda \boldsymbol{x}$ (and $\boldsymbol{x} \neq \boldsymbol{0}$), then $\boldsymbol{x}$ is an **eigenvector** of $A$ with **eigenvalue** $\lambda$ (just a number).

**Example 148.** Verify that $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$.

**Solution.** $A\boldsymbol{x} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} = 4\boldsymbol{x}$

Hence, $\boldsymbol{x}$ is an eigenvector of $A$ with eigenvalue $4$.

**Example 149.** Use your geometric understanding to find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

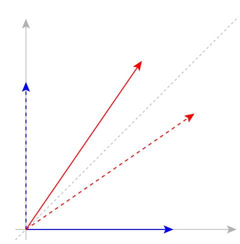


**Solution.** $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$

i.e. multiplication with $A$ is projection onto the $x$-axis.

- $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 1$.
- $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow \boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 0$.

**Example 150.** Use your geometric understanding to find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.



**Solution.** $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

i.e. multiplication with $A$ is reflection through the line $y = x$.

- $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 1$.
- $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightsquigarrow \boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = -1$.

## How to solve $A\boldsymbol{x} = \lambda\boldsymbol{x}$

$$\text{Key observation:} \quad \begin{aligned} & A\boldsymbol{x} = \lambda\boldsymbol{x} \\ \iff & A\boldsymbol{x} - \lambda\boldsymbol{x} = \mathbf{0} \\ \iff & (A - \lambda I)\boldsymbol{x} = \mathbf{0} \end{aligned}$$

This homogeneous system has a nontrivial solution if and only if $\det(A - \lambda I) = 0$.

**Recipe.** To find eigenvectors and eigenvalues of $A$.
(a) First, find the eigenvalues $\lambda$ by solving $\det(A - \lambda I) = 0$.
(b) Then, for each eigenvalue $\lambda$, find corresponding eigenvectors by solving $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$.

**Example 151.** Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

**Solution.**

- $A - \lambda I = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$
- $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = 0$

  $\Longrightarrow \lambda_1 = 2, \lambda_2 = 4$

  $\det(A - \lambda I) = \lambda^2 - 6\lambda + 8$ is the **characteristic polynomial** of $A$. Its roots are the eigenvalues of $A$.
- Find eigenvectors with eigenvalue $\lambda_1 = 2$:

  $A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

  Solutions to $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\boldsymbol{x} = \mathbf{0}$ have basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

  So: $\boldsymbol{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 2$.

  All other eigenvectors with $\lambda = 2$ are multiples of $\boldsymbol{x}_1$.
  $\operatorname{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ is the **eigenspace** for eigenvalue $\lambda = 2$.
- Find eigenvectors with eigenvalue $\lambda_2 = 4$:

  $A - \lambda_2 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

  Solutions to $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\boldsymbol{x} = \mathbf{0}$ have basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

  So: $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = 4$.

  The eigenspace for eigenvalue $\lambda = 4$ is $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.
- We check our answer:

  $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \overset{\checkmark}{=} 2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

  $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \overset{\checkmark}{=} 4\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

**Example 152.** Find the eigenvectors and the eigenvalues of $A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$.

**Solution.**

- The characteristic polynomial is:

  $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 & 3 \\ 0 & 6-\lambda & 10 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(6-\lambda)(2-\lambda)$

- $A$ has eigenvalues $2, 3, 6$.

| The eigenvalues of a triangular matrix are its diagonal entries. |
|---|

- $\lambda_1 = 2$:

  $(A - \lambda_1 I)\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 10 \\ 0 & 0 & 0 \end{bmatrix}\boldsymbol{x} = \mathbf{0} \implies \boldsymbol{x}_1 = \begin{bmatrix} 2 \\ -5/2 \\ 1 \end{bmatrix}$

- $\lambda_2 = 3$:

  $(A - \lambda_2 I)\boldsymbol{x} = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 10 \\ 0 & 0 & -1 \end{bmatrix}\boldsymbol{x} = \mathbf{0} \implies \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

- $\lambda_3 = 6$:

  $(A - \lambda_3 I)\boldsymbol{x} = \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 10 \\ 0 & 0 & -4 \end{bmatrix}\boldsymbol{x} = \mathbf{0} \implies \boldsymbol{x}_3 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

- In summary, $A$ has eigenvalues $2, 3, 6$ with corresponding eigenvectors $\begin{bmatrix} 2 \\ -5/2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix}$.

**Example 153. (Tuesday quiz!)** Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$.