Transition matrices

Powers of matrices can describe transition of a system.

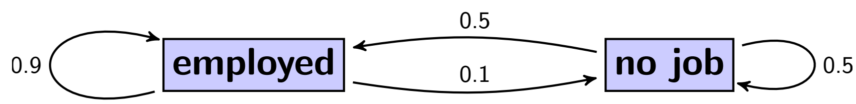
Example 1. (review)

- Fibonacci numbers F_n: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$
- $F_{n+1} = F_n + F_{n-1}$
- Hence: $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$

Example 2. Consider a fixed population of people with or without a job. Suppose that, each year, 50% of those unemployed find a job while 10% of those employed loose their job.

What is the unemployment rate in the long term equilibrium?

Solution.



x_t: employed at time t

y_t: unemployed at time t

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} =$$

The matrix $\begin{bmatrix} & \\ & \end{bmatrix}$ is a **Markov matrix**. Its columns add to 1 and it has no negative entries.

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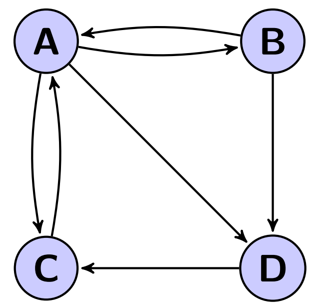
Page rank

Google's success is based on an algorithm to rank websites, the **Page rank**, named after Google founder Larry Page.

The basic idea is to determine how likely it is that a web user randomly gets to a given webpage. The webpages are then ranked by these probabilities.

Example 4. Suppose the internet consisted of only the four webpages A, B, C, D linked as in the following graph:



Imagine a surfer following these links at random.

For the probability $\mathrm{PR}_n(A)$ that she is at A (after n steps), we add:

- the probability that she was at B (at exactly one time step before), and left for A, (that's $\mathrm{PR}_{n-1}(B)\cdot\frac{1}{2}$)
- the probability that she was at C, and left for A,
- the probability that she was at D, and left for A.

- Hence: $\mathrm{PR}_n(A) = \mathrm{PR}_{n-1}(B)\cdot\frac{1}{2} + \mathrm{PR}_{n-1}(C)\cdot\frac{1}{1} + \mathrm{PR}_{n-1}(D)\cdot\frac{0}{1}$
- $\begin{bmatrix} \mathrm{PR}_n(A) \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} \mathrm{PR}_{n-1}(A) \\ \mathrm{PR}_{n-1}(B) \\ \mathrm{PR}_{n-1}(C) \\ \mathrm{PR}_{n-1}(D) \end{bmatrix}$
- The **PageRank vector** $\begin{bmatrix} \mathrm{PR}(A) \\ \mathrm{PR}(B) \\ \mathrm{PR}(C) \\ \mathrm{PR}(D) \end{bmatrix} = \begin{bmatrix} \mathrm{PR}_\infty(A) \\ \mathrm{PR}_\infty(B) \\ \mathrm{PR}_\infty(C) \\ \mathrm{PR}_\infty(D) \end{bmatrix}$ is the long-term equilibrium.

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Remark 5. In practical situations, the system might be too large for finding the eigenvector by elimination.

An alternative to elimination is the **power method**:

If T is an (acyclic and irreducible) Markov matrix, then for any $\boldsymbol{v}_0$ the vectors $T^n\boldsymbol{v}_0$ converge to an eigenvector with eigenvalue 1.

Here: $T = \begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} \mathrm{PR}(A) \\ \mathrm{PR}(B) \\ \mathrm{PR}(C) \\ \mathrm{PR}(D) \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.313 \\ 0.188 \end{bmatrix}$$

$T\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} =$

Remark 6.

- If all entries of T are positive, then the power method is guaranteed to work.
- In the context of PageRank, we can make sure that this is the case, by replacing T with
- Why does $T^n\boldsymbol{v}_0$ converge to an eigenvector with eigenvalue 1?

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