Eigenvectors and eigenvalues

Throughout, A will be an $n \times n$ matrix.

Definition 1. An **eigenvector** of A is a nonzero $\boldsymbol{x}$ such that

$$A\boldsymbol{x} = \lambda \boldsymbol{x} \quad \text{for some scalar } \lambda.$$

The scalar λ is the corresponding **eigenvalue**.

In words:

Example 2. Verify that $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$.

Solution.

Example 3. Use your geometric understanding to find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution.

Example 4. Use your geometric understanding to find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution.

Example 5. Let P be projection matrix onto the subspace V. What are the eigenvalues and eigenvectors of P?

Solution.

How to solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{\lambda}\boldsymbol{x}$

Key observation:

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

$$\Longleftrightarrow$$

$$\Longleftrightarrow$$

This has a nonzero solution $\Longleftrightarrow$

Recipe. To find eigenvectors and eigenvalues of A.

- First, find the eigenvalues λ using:

  λ is an eigenvalue of $A \iff \det(A - \lambda I) = 0$
- Then, for each eigenvalue, find the eigenvectors by

Example 6. Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Solution.

Example 7. Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}.$$

Solution.

Example 8. Find the eigenvectors and the eigenvalues of

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution.

The eigenvalues of a triangular matrix are its diagonal entries.

Example 9. Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

Solution.

Theorem 10. If $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are eigenvectors of A corresponding to different eigenvalues, then they are independent.

Why?

Review. $A\boldsymbol{x} = \lambda \boldsymbol{x}$

- To find the eigenvalues λ of A, we use $\det(A - \lambda I) = 0$.
  - $\det(A - \lambda I)$ is the **characteristic polynomial** of A.
  - If A is $n \times n$, then the characteristic polynomial has degree n.
- Then, for each eigenvalue, solve $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ to find the eigenvectors.

Two sources of trouble: eigenvalues can be

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Example 11. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Geometrically, what is the trouble?

Solution.

Example 12. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. What is the trouble?

Solution.

Practice problems

Example 13. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$.

Example 14. What are the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ -1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 4 \end{bmatrix}$?

No calculations!

Example 15. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}$.

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