Preparation problems for the discussion sections on October 7th and 9th

1. *Determine a basis for each of the following subspaces:*

(i) $H = \{\begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} : s, t \in \mathbb{R}\}$,

(ii) $K = \{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a - 3b + c = 0\}$,

(iii) $Col(\begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix})$,

(iv) $Nul(\begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix})$.

Solution:(i): Every vector in H is of them form

$$\begin{bmatrix} 4s \\ -3s \\ t \end{bmatrix} = s \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

So

$$H = \textbf{span}\{\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}.$$

Since these two vectors are linearly independent (they are not multiples of each other), $\{\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$ is a basis of H.

(ii) Every vector in K is of them form

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3b - c \\ b \\ c \\ d \end{bmatrix} = b \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus,

$$K = \textbf{span}(\{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\}).$$

I will leave it to you to check that these vectors are linearly independent and hence form a basis for K.

(iii) The matrix

$$A := \begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

is already of echelon form. Hence its pivot columns form a basis of $Col(A)$. Hence

$$\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$$

is basis of $Col(A)$.
(iv) Let A be as above. We bring A to reduced echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 - 3R2} \begin{bmatrix} 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Hence the free variables of $A\mathbf{x} = 0$ are x_2, x_5. We have $A\mathbf{x} = 0$ iff (that is, if and only if)

$$\begin{aligned} x_1 &= -2x_2 + 3x_5 \\ x_3 &= -x_5 \\ x_4 &= 0. \end{aligned}$$

Hence every vector $\mathbf{v} \in Nul(A)$ is of the form

$$\begin{bmatrix} -2x_2 + 3x_5 \\ x_2 \\ -x_5 \\ 0 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence

$$\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\}$$

form a basis of $Nul(A)$.

[Note: if you transform A into reduced echelon form and write $Nul(A)$ as span of coefficients of free variables (after substituting dependent variables in terms of free variables) then those vectors always will be linearly independent. So here there is no need to check linear independence, since we know they are in fact linearly independent.]

2. *Determine the dimension of $Nul(A)$ and $Col(A)$, where*

$$A := \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix}.$$

Solution: In order to determine the dimension of the two subspaces, we just have to determine the number of pivot columns and free variables of A. So we bring A to echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix} \xrightarrow{R2\to R2-R1, R3\to R3-2R1, R4\to R4-3R1} \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 0 & 0 & -3 & 6 & 0 \\ 0 & 0 & -9 & 18 & -7 \\ 0 & 0 & -9 & 18 & -15 \end{bmatrix}$$

$$\xrightarrow{R3\to R3-3R2, R4\to R4-3R2} \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 0 & 0 & -3 & 6 & 0 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & -15 \end{bmatrix}$$

$$\xrightarrow{R4\to R4-(15/7)R2} \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 0 & 0 & -3 & 6 & 0 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the echelon form of A has three pivots columns and two non-pivot columns. Hence $\dim Col(A)$ =number of pivot columns $= 3$ and $\dim Nul(A)$ =number of free variables $= 2$.

3. *Let A, B be two 4×3 matrices. Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ be the columns of A and let $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$ be the columns of B.*

(i) *Suppose that $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ is linearly independent. Find a basis for $Col(A)$ and describe $Nul(A)$.*

(ii) *Suppose that $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ is linearly independent and $\boldsymbol{b}_3 = 2\boldsymbol{b}_1 + 7\boldsymbol{b}_2$. Find a basis for $Col(B)$ and a basis for $Nul(B)$.*

Solution: (i) $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is a basis of $Col(A)$, because it spans $Col(A)$ (they are the columns of A!) and they are linearly independent (by assumption). Therefore, the rank of A is 3 and $Ax = 0$ has only the trivial solution. Hence $Nul(A) = \{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\}$.

(ii) $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis of $Col(B)$: by assumption it is linearly independent and $\mathbf{b}_3 = 2\mathbf{b}_1 + 7\mathbf{b}_2$, so we can show each column of B as a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$.Hence, we have $\mathbf{span}(\{\mathbf{b}_1, \mathbf{b}_2\}) = Col(B)$. Now let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be in $Nul(B)$. Then

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3.$$

Since $\mathbf{b}_3 = 2\mathbf{b}_1 + 7\mathbf{b}_2$, this happens iff

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3(2\mathbf{b}_1 + 7\mathbf{b}_2) = (x_1 + 2x_3)\mathbf{b}_1 + (x_2 + 7x_3)\mathbf{b}_2.$$

Since $\{\mathbf{b}_1, \mathbf{b}_2\}$ is linearly independent, the equation $y_1\mathbf{b}_1 + y_2\mathbf{b}_2 = 0$ has only the solution $y_1 = y_2 = 0$. Hence the above equation gives

$$0 = x_1 + 2x_3 = x_2 + 7x_3.$$

Hence every vector in $Nul(B)$ is of the form

$$\begin{bmatrix} -2x_3 \\ -7x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -7 \\ 1 \end{bmatrix}.$$

Hence $\{\begin{bmatrix} -2 \\ -7 \\ 1 \end{bmatrix}\}$ is a basis of $Nul(B)$.

[Here is a quicker way to do that: since the dimension of the column space and the null space add up to 3 (each column of B has to correspond to a free variable or contain a pivot), we know that $\dim Nul(B) = 1$. Elements of $Nul(B)$ correspond to linear relations between the columns of B. The fact that $2\mathbf{b}_1 + 7\mathbf{b}_2 - \mathbf{b}_3 = 0$, means that $[2, 7, -1]^T$ is in $Nul(B)$. Since the dimension is 1, this vector is a basis.]

4. *Let* $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *and let* $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.

(i) *Let* $\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. *Express* $\boldsymbol{v}$ *in terms of the basis* $\mathcal{B}$.

(ii) *Let* $\boldsymbol{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. *Express* $\boldsymbol{w}$ *in terms of the basis* $\mathcal{B}$.

(iii) *Let* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined such that* $T(\boldsymbol{v})$ *is expressing* $\boldsymbol{v}$ *in terms of the basis* $\mathcal{B}$. *(Convince yourself that this is a linear transformation.) Determine the matrix that represents* T *with respect to the standard basis of* $\mathbb{R}^2$.

Solution: (i) We need to solve

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 3 \end{array}\right] \xrightarrow{R2 \to R2 - R1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & 1 \end{array}\right]$$
$$\xrightarrow{R2 \to R2/(-2)} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -.5 \end{array}\right]$$
$$\xrightarrow{R1 \to R1 - R2} \left[\begin{array}{cc|c} 1 & 0 & 2.5 \\ 0 & 1 & -.5 \end{array}\right].$$

Hence

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - .5 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence $\mathbf{v}$ in terms of the basis $\mathcal{B}$ is $\begin{bmatrix} 2.5 \\ -.5 \end{bmatrix}$.

(ii) In contrast to the first part (but you can certainly proceed that way, if you don't see it), it is obvious that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence $\mathbf{w}$ in terms of the basis $\mathcal{B}$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(iii) Since

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = .5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + .5 \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

we have

$$T(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}.5\\.5\end{bmatrix}.$$

Since

$$\begin{bmatrix}0\\1\end{bmatrix} = .5\begin{bmatrix}1\\1\end{bmatrix} - .5\begin{bmatrix}1\\-1\end{bmatrix},$$

we have

$$T(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}.5\\-.5\end{bmatrix}.$$

Since $\begin{bmatrix}.5\\.5\end{bmatrix}$ and $\begin{bmatrix}.5\\-.5\end{bmatrix}$ in terms of the standard basis are $\begin{bmatrix}.5\\.5\end{bmatrix}$ and $\begin{bmatrix}.5\\-.5\end{bmatrix}$, respectively, the matrix that represents T with respect to the standard basis of $\mathbb{R}^2$ is

$$A = \begin{bmatrix}.5 & .5\\.5 & -.5\end{bmatrix}.$$

A remark. Since A is the matrix that represents T with respect to the standard basis of $\mathbb{R}^2$, we have

$$T(\mathbf{v}) = A\mathbf{v}.$$

For example,

$$T(\begin{bmatrix}2\\3\end{bmatrix}) = \begin{bmatrix}.5 & .5\\.5 & -.5\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}2.5\\-.5\end{bmatrix}.$$

Hence once you have calculated the matrix A, you have a simple way to express any vector $\mathbf{v}$ in terms of the basis $\mathcal{B}$.

5. *Let $L\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that*

$$L\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\8\\4\end{bmatrix}, \quad L\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\0\\1\end{bmatrix}.$$

What is $L\left(\begin{bmatrix}2\\1\end{bmatrix}\right)$?

Solution: Since L is linear, we get

$$L\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = L\left(2\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) = 2L\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + L\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = 2\begin{bmatrix}2\\8\\4\end{bmatrix} + \begin{bmatrix}3\\0\\1\end{bmatrix} = \begin{bmatrix}7\\16\\9\end{bmatrix}.$$

6. *Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation with*

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}5\\0\\1\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

(i) *Consider the basis $\mathcal{B}_1 = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$ of $\mathbb{R}^2$ and the basis $\mathcal{B}_2 = \{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\}$ of $\mathbb{R}^3$. Determine the matrix A which represents T with respect to the bases $\mathcal{B}_1$ and $\mathcal{B}_2$. Do you have $T(\boldsymbol{x}) = A\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^2$?*

(ii) *Consider the basis* $\mathcal{C}_1 := \{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\}$ *of* $\mathbb{R}^2$ *and the basis* $\mathcal{C}_2 = \{\begin{bmatrix}5\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\}$ *of* $\mathbb{R}^3$. *Determine the matrix* B *which represents* T *with respect to the bases* $\mathcal{C}_1$ *and* $\mathcal{C}_2$. *Do you have* $T(\boldsymbol{x}) = B\boldsymbol{x}$ *for all* $\boldsymbol{x} \in \mathbb{R}^2$*?*

Solution: For (i),

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\right) = \frac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) + \frac{1}{2}T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}5\\0\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{5}{2}\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$$

$$= \frac{5}{2}\begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

and

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(-\frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\right) = -\frac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) + \frac{1}{2}T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = -\frac{1}{2}\begin{bmatrix}5\\0\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-\frac{5}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}$$

$$= -\frac{5}{2}\begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Then

$$A = \begin{bmatrix}\frac{5}{2} & -\frac{5}{2}\\\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & -\frac{1}{2}\end{bmatrix}.$$

Yes, in this case $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$ (since both $\mathcal{B}_1$ and $\mathcal{B}_2$ are standard bases). For example,

$$T(\begin{bmatrix}1\\-1\end{bmatrix}) = \begin{bmatrix}\frac{5}{2} & -\frac{5}{2}\\\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}5\\0\\1\end{bmatrix}.$$

For (ii),

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}5\\0\\1\end{bmatrix} = 1\begin{bmatrix}5\\0\\1\end{bmatrix} + 0\begin{bmatrix}0\\1\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\end{bmatrix},$$

and

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix} = 0\begin{bmatrix}5\\0\\1\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Then

$$B = \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}.$$

No, $T(\mathbf{x}) \neq B\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^2$. For example,

$$\begin{bmatrix}5\\0\\1\end{bmatrix} = T(\begin{bmatrix}1\\-1\end{bmatrix}) \neq \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\-1\\0\end{bmatrix}.$$

[That's because, in order to use the matrix B for matrix multiplication, you have to convert to the bases with respect to which B has been constructed. Here, to compute $T([1,-1]^T)$, you have to first express the vector $[1,-1]^T$ with respect to the basis $\mathcal{C}_1$: this gives $[1,0]^T$. Now, you can use matrix multiplication: $B[1,0]^T = [1,0,0]^T$. Again, the output needs to be interpreted with respect to the correct basis, namely $\mathcal{C}_2$: this gives $[5,0,1]^T$. That's indeed $T([1,-1]^T)$.]

7. *Let $I\colon \mathbb{P}^3 \to \mathbb{P}^4$ be the integration linear transformation that maps p to*

$$\int_0^t p(t)dt.$$

Consider the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ of $\mathbb{P}^3$ and the basis $\mathcal{C} = \{1, t, t^2, t^3, t^4\}$ of $\mathbb{P}^4$. Determine the matrix which represents I with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.

Solution: Then

$$I(1) = t = 0 \cdot 1 + 1 \cdot t + 0 \cdot t^2 + 0 \cdot t^3 + 0 \cdot t^4,$$

and

$$I(t) = \frac{1}{2}t^2 = 0 \cdot 1 + 0 \cdot t + \frac{1}{2} \cdot t^2 + 0 \cdot t^3 + 0 \cdot t^4,$$

and

$$I(t^2) = \frac{1}{3}t^3 = 0 \cdot 1 + 0 \cdot t + 0 \cdot t^2 + \frac{1}{3} \cdot t^3 + 0 \cdot t^4,$$

and

$$I(t^3) = \frac{1}{4}t^4 = 0 \cdot 1 + 0 \cdot t + 0 \cdot t^2 + 0 \cdot t^3 + \frac{1}{4} \cdot t^4.$$

Hence the matrix A which represents I with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ is

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$