Preparation problems for the discussion sections on September 16th and 18th

1. (1) *Find a matrix E such that:*

$$E\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 - 2R_1 \\ R_3 \end{bmatrix}$$

Which matrix E^{-1} undoes the row operation implemented by E? What is $E^{-1}E$?

(2) *Find a matrix F such that:*

$$F\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_2 \\ R_1 \\ R_3 \end{bmatrix}$$

Which matrix F^{-1} undoes the row operation implemented by F? What is $F^{-1}F$?

(3) *Find a matrix G such that:*

$$G\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ 3R_2 \\ R_3 \end{bmatrix}$$

Which matrix G^{-1} undoes the row operation implemented by G? What is $G^{-1}G$?

Solution:

(1) $E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. To undo this row operation we have to replace $R_2 \to R_2 + 2R_1$.

So $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It is easy to check that $E^{-1}E = I$.

(2) $F = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. To undo this row operation we have to replace $R_2 \leftrightarrow R_1$. So

$F^{-1} = F = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It is easy to check that $F^{-1}F = I$.

(3) $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. To undo this row operation we have to replace $R_2 \to \frac{1}{3}R_2$. So

$G^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It is easy to check that $G^{-1}G = I$.

Note: each row operation on A is equivalent to multiplying A from the left by an invertible matrix.

2. *Consider the matrix:*

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}$$

Decompose the matrix A into LU, where L is a lower triangular matrix and U is an upper triangular matrix. Then use this factorization to solve:

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

That means, find a vector $\mathbf{c}$ in $\mathbb{R}^3$ such that:

$$L\mathbf{c} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

and then find a vector $\mathbf{x}$ in $\mathbb{R}^3$ such that:

$$U\mathbf{x} = \mathbf{c}$$

Solution: We start by bringing A to echelon form by multiplying A by elementary matrices. Let E be the matrix corresponding to subtracting row 1 three times from row 3, that is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Then

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}.$$

Since EA is already upper triangular, we set $U := EA$. Then $E^{-1}U = A$. Hence L is E^{-1}. To compute this explicitly, note that the inverse operation to subtracting row 1 from row 3 three times is adding row 1 three times to row 3. Hence

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

So the LU-decomposition of A is

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}.$$

[Note: the above steps include the maximum amount of details (not necessary for the exam). Can you see how to get L and U directly from the one row operation that is performed here?]

We now solve

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}.$$

We first solve

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

by forward substitution to get

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1. \end{bmatrix}$$

We then solve

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

by backward substitution to find

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

[Important note for the exam: time permitting, take a few moments to check that these values indeed solve the original problem.]

3. *Let* $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}$, *and* $U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$.

(1) *Show that* $A = LU$.
(2) *Let* A_i *be the* $i \times i$ *matrix introduced by the first* i *rows and the first* i *columns of* A, *for* $i = 1, 2, 3$. *What is an LU decomposition of* A_i, *for* $i = 1, 2, 3$*?*

Solution:

(1) Easy to check.
(2) An LU decomposition for A_i is $L_i U_i$ where L_i (respectively, U_i) is the matrix introduced by the first i rows and the first i columns of L (respectively, U).

4. *(more challenging) Let* A *and* B *be* $n \times n$ *matrices such that* $AB = I$.

(1) *What is the reduced echelon form of* A*?*
(2) *Show that* $BA = I$.

Solution:

(1) If the reduced echelon form of A has less than n pivot positions, then we get a row of zeros in the echelon form. So, there is a matrix F (namely, the product of all the elementary matrices used to reduce A) such that FA has a row of zeros. Therefore, $FAB = FI = F$ also has a row of zeros; but this is impossible since such a matrix F cannot possibly correspond to a reversible sequence of row operations.
So, the echelon form of A has exactly n pivot positions and we have exactly one pivot position in each row and in each column. These pivots are exactly on the main diagonal. Therefore, in the reduced echelon form every element on the main diagonal is 1 and every other element is equal to 0. Thus, the reduced echelon form of A is I.

(2) From the first part, there is a matrix F such that $FA = I$. We have:

$$F = FI = FAB = IB = B$$

So:

$$BA = FA = I$$

5. *Answer the following true-false questions. Explain why.*

(1) *If A is invertible then $A\mathbf{x} = \mathbf{0}$ has exactly one solution, $\mathbf{x} = \mathbf{0}$.*
(2) *If A is invertible then AB is also invertible.*
(3) *If A and B are invertible then $A + B$ is also invertible.*
(4) *If A is invertible then the reduced echelon form of A is equal to I.*

Solution:

(1) True, in this case $A\mathbf{x} = \mathbf{b}$ has exactly one solution, $\mathbf{x} = A^{-1}\mathbf{b}$.
(2) False, let $A = I$ and $B = 0$ (the zero matrix).
(3) False, let $A = I$ and $B = -I$.
(4) True, see the first part of problem (4).

6. *If $G = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$, find G^{-1}. Check that $G^{-1}G = I$.*

Solution: Use the following formula:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So, $A^{-1} = -\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$.

7. *Let $A = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. Use the Gauss-Jordan method to either find the inverse of A or to show that A is not invertible.*

Solution:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2\to R2-2R1} \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -3 & -2 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R3\to R3-R1} \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R3\to R3-R2/3} \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1/3 & -1/3 & 1 \end{array}\right]$$

We get a row of zeros in the left hand side, so it is not possible to transform the left hand side to the identity matrix. Thus, A is not invertible.

8. *Calculate the inverse of the matrix*

$$\begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Solution:

$$\left[\begin{array}{cccc|cccc} 2 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R3\leftrightarrow R4} \left[\begin{array}{cccc|cccc} 2 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R3\to R3-R4, R1\to R1+R4} \left[\begin{array}{cccc|cccc} 2 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R2\leftrightarrow R1} \left[\begin{array}{cccc|cccc} 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R2\to R2-2R1} \left[\begin{array}{cccc|cccc} 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & -2 & 0 & 1 & -2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R3\to R3-R1}\left[\begin{array}{cccc|cccc} 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & -2 & 0 & 1 & -2 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R2\to R2/3}\left[\begin{array}{cccc|cccc} 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2/3 & 0 & 1/3 & -2/3 & 1/3 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R1\to R1+R2}\left[\begin{array}{cccc|cccc} 1 & 0 & 1/3 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & -2/3 & 0 & 1/3 & -2/3 & 1/3 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R3\to R3-2R2}\left[\begin{array}{cccc|cccc} 1 & 0 & 1/3 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & -2/3 & 0 & 1/3 & -2/3 & 1/3 & 0 \\ 0 & 0 & 4/3 & 0 & -2/3 & 1/3 & -5/3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R3\to 3R3/4}\left[\begin{array}{cccc|cccc} 1 & 0 & 1/3 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & -2/3 & 0 & 1/3 & -2/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & -2/4 & 1/4 & -5/4 & 3/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R2\to R2+2/3R3}\left[\begin{array}{cccc|cccc} 1 & 0 & 1/3 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 0 & -2/4 & 1/4 & -5/4 & 3/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R1\to R1-1/3R3}\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/2 & 1/4 & 3/4 & -1/4 \\ 0 & 1 & 0 & 0 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 0 & -2/4 & 1/4 & -5/4 & 3/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right].$$

So the inverse matrix is:

$$\begin{bmatrix} 1/2 & 1/4 & 3/4 & -1/4 \\ 0 & -1/2 & -1/2 & 1/2 \\ -2/4 & 1/4 & -5/4 & 3/4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

9. *Consider the equation:*

$$-\frac{d^2u}{dx^2} = 4\pi^2 \sin 2\pi x, \quad u(0) = u(1) = 0$$

(1) *Write down the* 3 *by* 3 *matrix equation with* $h = \frac{1}{4}$.

(2) *Solve for* u_1, u_2, u_3 *and find their error in comparison with the true solution* $u = \sin 2\pi x$ *at* $x = \frac{1}{4}$, $x = \frac{1}{2}$, *and* $x = \frac{3}{4}$.

Solution:

(1) We have:

$$u_0 = u(0) = 0,\ u_1 = u(\frac{1}{4}),\ u_2 = u(\frac{2}{4}),\ u_3 = u(\frac{3}{4}),\ u_4 = u(1) = 0$$

From the equation we get:

$$u_{i-1} - 2u_i + u_{i+1} = (\frac{1}{4})^2 4\pi^2 \sin(2\pi \frac{i}{4}) \quad \text{for} \quad i = 1, 2, 3$$

So, the matrix equation is:

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = - \begin{bmatrix} (\frac{1}{4})^2 4\pi^2 \sin(2\pi\frac{1}{4}) \\ (\frac{1}{4})^2 4\pi^2 \sin(2\pi\frac{2}{4}) \\ (\frac{1}{4})^2 4\pi^2 \sin(2\pi\frac{3}{4}) \end{bmatrix} = \begin{bmatrix} -\frac{\pi^2}{4} \\ 0 \\ \frac{\pi^2}{4} \end{bmatrix}$$

(2) We have:

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & -\frac{\pi^2}{4} \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & \frac{\pi^2}{4} \end{array}\right] \xrightarrow{R1 \to -1/2 R1} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & \frac{\pi^2}{8} \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & \frac{\pi^2}{4} \end{array}\right]$$

$$\xrightarrow{R2 \to R2 - R1} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & \frac{\pi^2}{8} \\ 0 & -\frac{3}{2} & 1 & -\frac{\pi^2}{8} \\ 0 & 1 & -2 & \frac{\pi^2}{4} \end{array}\right] \xrightarrow{R2 \to -2/3 R2} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & \frac{\pi^2}{8} \\ 0 & 1 & -\frac{2}{3} & \frac{\pi^2}{12} \\ 0 & 1 & -2 & \frac{\pi^2}{4} \end{array}\right]$$

$$\xrightarrow{R3 \to R3 - R2} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & \frac{\pi^2}{8} \\ 0 & 1 & -\frac{2}{3} & \frac{\pi^2}{12} \\ 0 & 0 & -\frac{4}{3} & \frac{\pi^2}{6} \end{array}\right] \xrightarrow{R1 \to R1 + 1/2 R2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & \frac{\pi^2}{6} \\ 0 & 1 & -\frac{2}{3} & \frac{\pi^2}{12} \\ 0 & 0 & -\frac{4}{3} & \frac{\pi^2}{6} \end{array}\right]$$

$$\xrightarrow{R3 \to -3/4 R3} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & \frac{\pi^2}{6} \\ 0 & 1 & -\frac{2}{3} & \frac{\pi^2}{12} \\ 0 & 0 & 1 & -\frac{\pi^2}{8} \end{array}\right] \xrightarrow{R2 \to R2 + 2/3 R3} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & \frac{\pi^2}{6} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{\pi^2}{8} \end{array}\right]$$

$$\xrightarrow{R1 \to R1 + 1/3 R3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{\pi^2}{8} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{\pi^2}{8} \end{array}\right]$$

[Note: depending on your taste, you can save some ink by only computing an echelon form and then solving by backward substitution. Also, you could do and use an LU decomposition; since we definitely only deal with a single right-hand side here, that is up to you.]

So the solution is:

$$\begin{bmatrix} \frac{\pi^2}{8} \\ 0 \\ -\frac{\pi^2}{8} \end{bmatrix}$$

Finally, the error is:

$$\begin{bmatrix} \frac{\pi^2}{8} - 1 \\ 0 \\ -\frac{\pi^2}{8} + 1 \end{bmatrix} \approx \begin{bmatrix} 0.234 \\ 0 \\ -0.234 \end{bmatrix}$$