Preparation problems for the discussion sections on September 2nd and 4th

1. *For the following systems determine*

(1) *the augmented matrix,*
(2) *an echelon form of the matrix,*
(3) *the reduced echelon form of the matrix,*
(4) *whether the system is consistent,*
(5) *the parametric description of the set of solutions,*
(6) *how many solution the system has,*
(7) *the geometric interpretation of the set of solutions.*

System A:

$$\begin{aligned} x_2 &= 3 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

System B:

$$\begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 + 2x_2 &= 6 \end{aligned}$$

System C:

$$\begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 + 2x_2 &= 7 \end{aligned}$$

Solution: **System A:** For (1): The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 2 & 4 \end{array}\right].$$

For (2): We reduce it to echelon form as follows:

$$\left[\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \end{array}\right].$$

Note that this is not the reduced echelon form of the augmented matrix, but it is enough to answer (4). The system is consistent, because in echelon form there is no row of the form

$$\left[\begin{array}{cc|c} 0 & 0 & x \end{array}\right],$$

where x is non-zero.

For (3): To get to the reduced echelon form:

$$\left[\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{R1 \leftrightarrow R1 - 2R2} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \end{array}\right].$$

For (5): By (3), we get that System A has the same solutions as

$$\begin{aligned} x_1 &= -2 \\ x_2 &= 3. \end{aligned}$$

Hence the parametric description of the set of solutions is simply

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

For (6): The system has exactly one solution.
For (7): The set of solutions is a point in $\mathbb{R}^2$.

System B: For (1): The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 6 \end{array}\right].$$

For (2): We reduce it to echelon form as follows:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 6 \end{array}\right] \xrightarrow{R2 \leftrightarrow R2-2R1} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right].$$

Note that is also the reduced echelon form of the augmented matrix, answering (3).
For (4): The system is consistent, because in echelon form there is no row of the form

$$\left[\begin{array}{cc|c} 0 & 0 & x \end{array}\right],$$

where x is non-zero.
For (5): By (3), we get that System B has the same solutions as

$$x_1 + x_2 = 3.$$

Hence the parametric description of the set of solutions is simply

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 - x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

For (6): The system has infinitely many solution, because it is consistent and has a free variable (x_2).
For (7): The set of solutions is a line in $\mathbb{R}^2$.

System C: For (1): The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 7 \end{array}\right].$$

For (2): We reduce it to echelon form as follows:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 7 \end{array}\right] \xrightarrow{R2 \leftrightarrow R2-2R1} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right].$$

For (3): To get to the reduced echelon form:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{R1 \leftrightarrow R1-3R2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

For (4): The system is inconsistent, because in echelon form there is a row of the form

$$\left[\begin{array}{cc|c} 0 & 0 & x \end{array}\right],$$

where x is non-zero.
For (5): By (4), we get that System C is inconsistent and hence there is no solution.
For (6): No solution.
For (7): The set of solutions is empty.

2. *Some question to check your understanding:*

a) *What is the largest possible number of pivots a 4×6 matrix can have? Why?*
b) *What is the largest possible number of pivots a 6×4 matrix can have? Why?*
c) *How many solutions does a consistent linear system of 3 equations and 4 unknowns have? Why?*
d) *Suppose the coefficient matrix corresponding to a linear system is 4×6 and has 3 pivot columns. How many pivot columns does the augmented matrix have if the linear system is inconsistent?*

Solution: For (a): 4. Each row can have at most one leading entry and hence at most one pivot.
For (b): Again 4. Each column has at most one pivot.
For (c): The augmented matrix of this system has 3 rows and 4 columns. Hence it has at most 3 pivots. So one variable cannot correspond to a pivot column. Hence the system has one free variable. Any consistent system with a free variable has infinitely many solutions.
For (d): 4. In order to be inconsistent, the augmented matrix in echelon form has a row $\left[\begin{array}{ccc|c} 0 & \dots & 0 & x \end{array}\right]$, where $x \neq 0$. This row gives the augmented matrix a forth pivot.

3. *Find the parametric description of the set of solutions of*

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 4 \\ x_1 + 4x_2 - 8x_3 &= 7 \\ -3x_1 - 7x_2 + 9x_3 &= -6 \end{aligned}$$

Solution: We bring the augmented matrix to reduced echelon form:

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{array}\right] \xrightarrow{R2 \to R2 - R1, R3 \to R3 + 3R1} \left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{array}\right]$$
$$\xrightarrow{R3 \to R3 - 2R2} \left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\xrightarrow{R1 \to R1 - 3R2} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence the parametric description of the set of solutions is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 - 4x_3 \\ 3 + 3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}.$$

4. *For which values of h_1 and h_2 is the following system consistent?*

$$x_1 = h_1$$
$$x_2 = 5$$
$$x_1 + 2x_2 = h_2$$

Solution: We bring the augmented matrix to echelon form:

$$\left[\begin{array}{cc|c} 1 & 0 & h_1 \\ 0 & 1 & 5 \\ 1 & 2 & h_2 \end{array}\right] \xrightarrow{R3 \to R3 - R1} \left[\begin{array}{cc|c} 1 & 0 & h_1 \\ 0 & 1 & 5 \\ 0 & 2 & h_2 - h_1 \end{array}\right] \xrightarrow{R3 \to R3 - 2R2} \left[\begin{array}{cc|c} 1 & 0 & h_1 \\ 0 & 1 & 5 \\ 0 & 0 & h_2 - h_1 - 10 \end{array}\right].$$

Hence the augmented matrix contains a row of the form $\left[\begin{array}{cc|c} 0 & 0 & x \end{array}\right]$, where x is non-zero, if and only if $h_2 - h_1 - 10 \neq 0$. Hence the system is consistent if and only if $h_2 - h_1 = 10$.

5. *Show that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.*

solution: We can replace $R_i \leftrightarrow R_j$ with the following sequence of row operations: $(i \neq j)$

$$R_i \leftrightarrow R_i + R_j,\ R_j \leftrightarrow R_j - R_i,\ R_j \leftrightarrow -R_j,\ R_i \leftrightarrow R_i - R_j$$

6. *Let $A = [a_{ij}]_{3\times 4}$, and let $B = [b_{ij}]_{3\times 4}$ be an echelon form of A.*

(1) *Is it true that, if $a_{11} = 0$, then $b_{11} = 0$?*
(2) *Is it true that, if A has a column of zeros, then B also has a column of zeros?*
(3) *Suppose B has a row of zeros. What can you say about rows of A? (Explain.)*
(4) *Suppose we form a new matrix using some columns of A, let's say the first and the third column. What is an echelon form corresponding to this new matrix?*

solution:

(1) False, consider $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.
(2) True, since row operations do not change a column of zeros.
(3) Each row of B is a linear combination of rows of A with at least one non-zero coefficient. Therefore, a linear combination of rows of A with at least one non-zero coefficient is equal to zero. (In language that we have not yet learned, this means that the rows of A are "linearly dependent".)
(4) The matrix consisting of the first and the third column of B has the same reduced echelon form as the new matrix introduced in the problem. So, instead of transforming the new matrix into echelon form you can transform the matrix consisting of the first and the third column of B into echelon form.
(The example $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and $B = A$ shows that just taking first and third column does not necessarily result in an echelon form.)