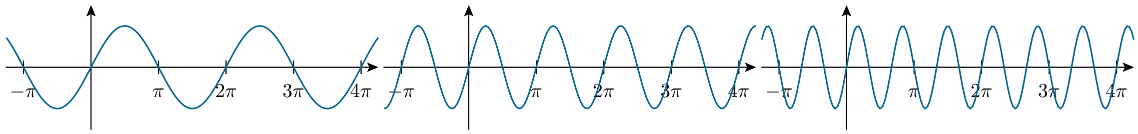
Review

- An inner product for 2π-periodic functions:

  $\langle f, g \rangle = \int_0^{2\pi} f(x) g(x) \mathrm{d}x$ (in $\mathbb{R}^n$: $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = v_1 w_1 + \ldots + v_n w_n$)

- $1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots$ are orthogonal



- An expansion in that basis is a **Fourier series**:

$$f(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \cdots$$

where

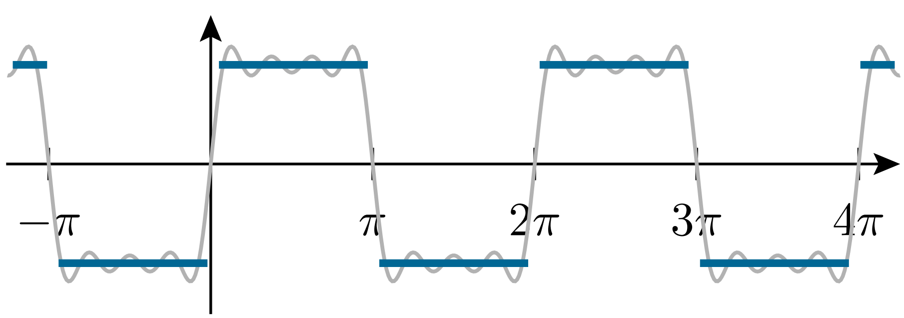
$$a_k = \frac{\langle f(x), \cos(kx) \rangle}{\langle \cos(kx), \cos(kx) \rangle} = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx) \mathrm{d}x,$$

$$b_k = \frac{\langle f(x), \sin(kx) \rangle}{\langle \sin(kx), \sin(kx) \rangle} = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx) \mathrm{d}x,$$

$$a_0 = \frac{\langle f(x), 1 \rangle}{\langle 1, 1 \rangle} = \frac{1}{2\pi} \int_0^{2\pi} f(x) \mathrm{d}x.$$

Example 1.

$$\text{blue function} = \frac{4}{\pi} \left(\sin(x) + \frac{1}{3} \sin(3x) + \frac{1}{5} \sin(5x) + \frac{1}{7} \sin(7x) + \ldots \right)$$



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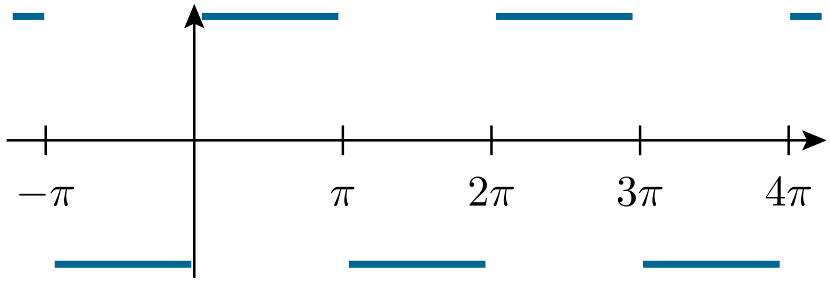
Example 2. Find the Fourier series of the 2π-periodic function $f(x)$ defined by

$$f(x) = \begin{cases} -1, & \text{for } x \in (-\pi, 0), \\ +1, & \text{for } x \in (0, \pi). \end{cases}$$



Solution. Note that $\int_0^{2\pi}$ and $\int_{-\pi}^{\pi}$ are the same here. (why?!)

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = 0 \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x \\
&= \frac{1}{\pi}\left[-\int_{-\pi}^{0} \cos(nx)\mathrm{d}x + \int_0^{\pi} \cos(nx)\mathrm{d}x\right] = 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x \\
&= \frac{1}{\pi}\left[-\int_{-\pi}^{0} \sin(nx)\mathrm{d}x + \int_0^{\pi} \sin(nx)\mathrm{d}x\right] \\
&= \frac{2}{\pi}\left[\int_0^{\pi} \sin(nx)\mathrm{d}x\right] \\
&= \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi} \\
&= \frac{2}{\pi n}[1 - \cos(n\pi)] \\
&= \frac{2}{\pi n}[1 - (-1)^n] = \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}
\end{aligned}$$

In conclusion,

$$f(x) = \frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \ldots\right).$$

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Determinants

For the next few lectures, all matrices are square!

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The **determinant** of

- a 2×2 matrix is $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc$,
- a 1×1 matrix is $\det([\, a \,]) = a$.

Goal: A is invertible $\iff$ $\det(A) \neq 0$

We will write both $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ and $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ for the determinant.

Definition 3. The **determinant** is characterized by:

- the normalization $\det I = 1$,
- and how it is affected by elementary row operations:
  - **(replacement)** Add one row to a multiple of another row.
    Does not change the determinant.
  - **(interchange)** Interchange two rows.
    Reverses the sign of the determinant.
  - **(scaling)** Multiply all entries in a row by s.
    Multiplies the determinant by s.

Example 4. Compute $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{vmatrix}$.

Solution.

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{vmatrix} \overset{R2\to\frac{1}{2}R2}{=} 2\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{vmatrix} \overset{R3\to\frac{1}{7}R3}{=} 14\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 14$$

Example 5. Compute $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{vmatrix}$.

Solution.

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{vmatrix} \overset{R2\to\frac{1}{2}R2}{=} 2\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 7 \end{vmatrix} \overset{R3\to\frac{1}{7}R3}{=} 14\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\overset{\substack{R1\to R1-3R3 \\ R2\to R2-2R3}}{=} 14\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \overset{R1\to R1-2R2}{=} 14\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 14$$

The determinant of a triangular matrix is the product of the diagonal entries.

Example 6. Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$.

Solution.

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} \overset{\substack{R2 \to R2 - 3R1 \\ R3 \to R3 - 2R1}}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & -4 & 1 \end{vmatrix}$$

$$\overset{R3 \to R3 - \frac{4}{7}R2}{=} \begin{vmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & -\frac{1}{7} \end{vmatrix}$$

$$= 1 \cdot (-7) \cdot \left(-\frac{1}{7}\right) = 1$$

Example 7. Discover the formula for $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Solution.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{R2 \to R2 - \frac{c}{a}R1}{=} \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{vmatrix} = a\left(d - \frac{c}{a}b\right) = ad - bc$$

Example 8. Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$.

Solution.

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix} \overset{R4 \to R4 - \frac{3}{2}R3}{=} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & \frac{7}{2} \end{vmatrix} = 1 \cdot 2 \cdot 2 \cdot \frac{7}{2} = 14$$

The following important properties follow from the behaviour under row operations.

- $\det(A) = 0 \iff A$ is not invertible

  Why? Because $\det(A) = 0$ if only if, in an echelon form, a diagonal entry is zero (that is, a pivot is missing).

- $\det(AB) = \det(A)\det(B)$
- $\det(A^{-1}) = \frac{1}{\det(A)}$
- $\det(A^T) = \det(A)$

Example 9. Recall that $AB = \mathbf{0}$, then it does not follow that $A = \mathbf{0}$ or $B = \mathbf{0}$. However, show that $\det(A) = 0$ or $\det(B) = 0$.

Solution. Follows from $\det(AB) = \det(\mathbf{0}) = 0$,
and $\det(AB) = \det(A)\det(B)$.

A "bad" way to compute determinants

Example 10. Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution. We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} + & \blacksquare & \blacksquare \\ \blacksquare & -1 & 2 \\ \blacksquare & 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} \blacksquare & - & \blacksquare \\ 3 & \blacksquare & 2 \\ 2 & \blacksquare & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} \blacksquare & \blacksquare & + \\ 3 & -1 & \blacksquare \\ 2 & 0 & \blacksquare \end{vmatrix}$$

$$\overset{\text{i.e.}}{=} 1 \cdot \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = 1 \cdot (-1) - 2 \cdot (-1) + 0 = 1$$

Each term in the cofactor expansion is ± 1 times an entry times a smaller determinant (row and column of entry deleted).

The ± 1 is assigned to each entry according to $\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$.

Solution. We expand by the second column:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -2 \cdot \begin{vmatrix} \blacksquare & - & \blacksquare \\ 3 & \blacksquare & 2 \\ 2 & \blacksquare & 1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 1 & \blacksquare & 0 \\ \blacksquare & + & \blacksquare \\ 2 & \blacksquare & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & \blacksquare & 0 \\ 3 & \blacksquare & 2 \\ \blacksquare & - & \blacksquare \end{vmatrix}$$

$$= -2 \cdot (-1) + (-1) \cdot 1 - 0 = 1$$

Solution. We expand by the third column:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0 \cdot \begin{vmatrix} \blacksquare & \blacksquare & + \\ 3 & -1 & \blacksquare \\ 2 & 0 & \blacksquare \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 2 & \blacksquare \\ \blacksquare & \blacksquare & - \\ 2 & 0 & \blacksquare \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 & \blacksquare \\ 3 & -1 & \blacksquare \\ \blacksquare & \blacksquare & + \end{vmatrix}$$

$$= 0 - 2 \cdot (-4) + 1 \cdot (-7) = 1$$

Practice problems

Problem 1. Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$.

Solution. The final answer should be -10.

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