The inhomogeneous heat equation

We next indicate that we can similarly solve the inhomogeneous heat equation (with inhomogeneous boundary conditions).

Comment. We indicated earlier that

$$u_t = k u_{xx}$$
 (PDE)
 $u(0,t) = a, \quad u(L,t) = b$ (BC)
 $u(x,0) = f(x), \quad x \in (0,L)$ (IC)

can be solved by realizing that Ax + B solves (PDE).

Indeed, let $v(x) = a + \frac{b-a}{L}x$ (so that v(0) = a and v(L) = b). We then look for a solution of the form u(x,t) = v(x) + w(x,t). Note that u(x,t) solves (PDE)+(BC)+(IC) if and only if w(x,t) solves:

$w_t = k w_{xx}$	(PDE)
w(0,t) = 0, w(L,t) = 0	(BC^*)
$w(x,0) = f(x) - v(x), x \in (0,L)$	(IC)

This is the (homogeneous) heat equation that we know how to solve.

v(x) is called the **steady-state solution** (it does not depend on time!) and w(x,t) the **transient solution** (note that w(x,t) and its partial derivatives tend to zero as $t \to \infty$ because of the boundary conditions (BC*)).

Example 167. Consider the heat flow problem: $\begin{array}{c} u_t=3u_{xx}+4x^2 & (\mathrm{PDE})\\ u(0,t)=1, \quad u_x(3,t)=-5 & (\mathrm{BC})\\ u(x,0)=f(x), \quad x\in(0,3) & (\mathrm{IC}) \end{array}$

Determine the steady-state solution and spell out equations characterizing the transient solution.

Solution. We look for a solution of the form u(x,t) = v(x) + w(x,t), where v(x) is the steady-state solution and where w(x,t) is the transient solution which (together with its derivatives) tends to zero as $t \to \infty$.

- Plugging into (PDE), we get $w_t = 3v'' + 3w_{xx} + 4x^2$. Letting $t \to \infty$, this becomes $0 = 3v'' + 4x^2$. Note that this also implies that $w_t = 3w_{xx}$.
- Plugging into (BC), we get v(0) + w(0,t) = 1 and $v'(3) + w_x(3,t) = -5$. Letting $t \to \infty$, these become v(0) = 1 and v'(3) = -5.
- Solving the ODE $0 = 3v'' + 4x^2$, we find

$$v(x) = \iint -\frac{4}{3}x^2 dx dx = \int \left(-\frac{4}{9}x^3 + C\right) dx = -\frac{1}{9}x^4 + Cx + D.$$

The boundary conditions v(0) = 1 and v'(3) = -5 imply D = 1 and $-\frac{4}{9} \cdot 3^3 + C = -5$ (so that C = 7). In conclusion, the steady-state solution is $v(x) = -\frac{1}{9}x^4 + 1 + 7x$.

On the other hand, the transient solution w(x,t) is characterized as the unique solution to:

This homogeneous heat flow problem can now be solved using separation of variables.

Example 168. For $t \ge 0$ and $x \in [0, 4]$, consider the heat flow problem: $\begin{array}{l} u_t = 2u_{xx} + e^{-x/2} \\ u_x(0, t) = 3 \\ u(4, t) = -2 \\ u(x, 0) = f(x) \end{array}$

Determine the steady-state solution and spell out equations characterizing the transient solution. Solution. We look for a solution of the form u(x,t) = v(x) + w(x,t), where v(x) is the steady-state solution and where the transient solution w(x,t) tends to zero as $t \to \infty$ (as do its derivatives).

- Plugging into (PDE), we get $w_t = 2v'' + 2w_{xx} + e^{-x/2}$. Letting $t \to \infty$, this becomes $0 = 2v'' + e^{-x/2}$.
- Plugging into (BC), we get $w_x(0,t) + v'(0) = 3$ and w(4,t) + v(4) = -2. Letting $t \to \infty$, these become v'(0) = 3 and v(4) = -2.
- Solving the ODE $0 = 2v'' + e^{-x/2}$, we find

$$v(x) = \iint -\frac{1}{2}e^{-x/2} dx dx = \int (e^{-x/2} + C) dx = -2e^{-x/2} + Cx + D$$

The boundary conditions v'(0) = 3 and v(4) = -2 imply C = 2 and $-2e^{-2} + 8 + D = -2$. In conclusion, the steady-state solution is $v(x) = -2e^{-x/2} + 2x - 10 + 2e^{-2}$.

On the other hand, the transient solution w(x,t) is characterized as the unique solution to:

$$w_t = 2w_{xx}$$

 $w_x(0,t) = 0, \quad w(4,t) = 0$
 $w(x,0) = f(x) - v(x)$

Note. We know how to solve this homogeneous heat equation using separation of variables.

Steady-state temperature: The Laplace equation

(2D and 3D heat equation) In higher dimensions, the heat equation takes the form $u_t = k(u_{xx} + u_{yy})$ or $u_t = k(u_{xx} + u_{yy} + u_{zz})$.

The heat equation is often written as $u_t = k \Delta u$ where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ (2D) or $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (3D) is the Laplace operator you may know from Calculus III. Other notations. $\Delta u = \operatorname{div} \operatorname{grad} u = \nabla \cdot \nabla u = \nabla^2 u$

If temperature is steady, then $u_t = 0$. Hence, the steady-state temperature u(x, y) must satisfy the PDE $u_{xx} + u_{yy} = 0$.

(Laplace equation, 2D)

 $u_{xx} + u_{yy} = 0$

Comment. The Laplace equation is so important that its solutions have their own name: harmonic functions. It is also known as the "potential equation"; satisfied by electric/gravitational potential functions. (More generally, such potentials, if not in the vacuum, satisfy the Poisson equation $u_{xx} + u_{yy} = f(x, y)$, the inhomogeneous version of the Laplace equation.)

Recall from Calculus III (if you have taken that class) that the gradient of a scalar function f(x, y) is the vector field $\mathbf{F} = \operatorname{grad} f = \nabla f = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix}$. One says that \mathbf{F} is a gradient field and f is a potential function for \mathbf{F} (for instance, \mathbf{F} could be a gravitational field with gravitational potential f).

The divergence of a vector field $G = \begin{bmatrix} g(x, y) \\ h(x, y) \end{bmatrix}$ is div $G = g_x + h_y$. One also writes div $G = \nabla \cdot G$.

The gradient field of a scalar function f is divergence-free if and only if f satisfies the Laplace equation $\Delta f = 0$.

One way to describe a unique solution to the Laplace equation within a region is by specifying the values of u(x, y) along the boundary of that region.

This is particularly natural for steady-state temperatures profiles of a region R. The Laplace equation governs how temperature behaves inside the region but we need to also prescribe the temperature on the boundary.

The PDE with such a boundary condition is called a Dirichlet problem:

(Dirichlet problem) $u_{xx} + u_{yy} = 0$ within region Ru(x, y) = f(x, y) on boundary of R

In general. A Dirichlet problem consists of a PDE, that needs to hold within a region R, and prescribed values on the boundary of that region ("Dirichlet boundary conditions").