Final Exam – Practice

MATH 332 — Differential Equations II
Final Exam: Monday, May 5, 2025

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. The final exam will be comprehensive, that is, it will cover the material of the whole semester.

(a) Do the practice problems for both midterms.

(b) Retake both midterm exams.

(c) Do the problems below. (Solutions are posted.)

Problem 2. For $t \geqslant 0$ and $x \in [0, 4]$, consider the heat flow problem:
$$\begin{aligned} u_t &= 2u_{xx} + e^{-x/2} \\ u_x(0,t) &= 3 \\ u(4,t) &= -2 \\ u(x,0) &= f(x) \end{aligned}$$

Determine the steady-state solution and spell out equations characterizing the transient solution.

Solution. We look for a solution of the form $u(x,t) = v(x) + w(x,t)$, where $v(x)$ is the steady-state solution and where the transient solution $w(x,t)$ tends to zero as $t \to \infty$ (as do its derivatives).

- Plugging into (PDE), we get $w_t = 2v'' + 2w_{xx} + e^{-x/2}$. Letting $t \to \infty$, this becomes $0 = 2v'' + e^{-x/2}$.
- Plugging into (BC), we get $w_x(0,t) + v'(0) = 3$ and $w(4,t) + v(4) = -2$.
  Letting $t \to \infty$, these become $v'(0) = 3$ and $v(4) = -2$.
- Solving $0 = 2v'' + e^{-x/2}$, we find
  $$v(x) = \iint -\frac{1}{2} e^{-x/2} \mathrm{d}x \mathrm{d}x = \int e^{-x/2} \mathrm{d}x + C = -2e^{-x/2} + Cx + D.$$
  The boundary conditions $v'(0) = 3$ and $v(4) = -2$ imply $C = 2$ and $-2e^{-2} + 8 + D = -2$.
  and therefore the steady-state solution $v(x) = -2e^{-x/2} + 2x - 10 + 2e^{-2}$.

On the other hand, the transient solution $w(x,t)$ is characterized as the unique solution to:

$$\begin{aligned} &w_t = 2w_{xx} \\ &w_x(0,t) = 0, \quad w(4,t) = 0 \\ &w(x,0) = f(x) - v(x) \end{aligned}$$

Note. We know how to solve this homogeneous heat equation (see Problem 6) using separation of variables.

Problem 3. Using a step size of $h = \frac{1}{3}$, discretize the Dirichlet problem
$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ u(x,0) &= 3 \\ u(x,1) &= 5 \\ u(0,y) &= 1 \\ u(1,y) &= 2 \end{aligned}$$
where $x \in (0,1)$ and $y \in (0,1)$.

Spell out a system of linear equations for the resulting lattice points. Do not solve that system.

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(Note that, for the Dirichlet problem as well as for our discretization, it doesn't matter that the boundary conditions aren't well-defined at the corners.)

Solution. Note that our region is a square with side lengths 1 (in both x and y directions). We write $u_{m,n} = u(mh, nh)$. Make a sketch!

$$\begin{array}{cccc} & 5 & 5 & \\ 1 & u_{1,2} & u_{2,2} & 2 \\ 1 & u_{1,1} & u_{2,1} & 2 \\ & 3 & 3 & \end{array}$$

We need to determine equations for the four unknowns $u_{1,1}, u_{2,1}, u_{1,2}, u_{2,2}$.

If we approximate $u_{xx} + u_{yy}$ by $\frac{1}{h^2}[u(x-h,y) + u(x+h,y) + u(x,y-h) + u(x,y+h) - 4u(x,y)]$ then, in terms of our lattice points, the equation $u_{xx} + u_{yy} = 0$ translates into

$$u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - 4u_{m,n} = 0.$$

We get one such equation for each of the four unknowns (where the unknown is multiplied by -4). For instance, the equation for $u_{2,1}$ is

$$u_{1,1} + \underbrace{u_{3,1}}_{=2} + \underbrace{u_{2,0}}_{=3} + u_{2,2} - 4u_{2,1} = 0.$$

Spelling out these equation in matrix-vector form (the above equation for $u_{2,1}$ is the second row), we obtain:

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{1,2} \\ u_{2,2} \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ -6 \\ -7 \end{bmatrix}$$

Problem 4. Consider the Laplace equation $u_{xx} + u_{yy} = 0$ on the polygonal region with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(2,1)$, $(2,2)$, $(0,2)$. Suppose that $u(0,y) = 5$ for $y \in (0,2)$ and that $u(x,y) = 7$ for all other points (x,y) on the boundary of the region. Discretize this Dirichlet problem using a step size of $h = \frac{1}{2}$.

Spell out a system of linear equations for the resulting lattice points. Do not solve that system.

Solution. We write $u_{m,n} = u(mh, nh)$. Make a sketch!

$$\begin{array}{ccccc} & 7 & 7 & 7 & \\ 5 & u_{1,3} & u_{2,3} & u_{3,3} & 7 \\ 5 & u_{1,2} & 7 & 7 & \\ 5 & u_{1,1} & 7 & & \\ & 7 & & & \end{array}$$

If we approximate $u_{xx} + u_{yy}$ by $\frac{1}{h^2}[u(x-h,y) + u(x+h,y) + u(x,y-h) + u(x,y+h) - 4u(x,y)]$ then, in terms of our lattice points, the equation $u_{xx} + u_{yy} = 0$ translates into

$$u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - 4u_{m,n} = 0.$$

Spelling out these equation in matrix-vector form, we obtain:

$$\begin{bmatrix} -4 & 1 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} -19 \\ -12 \\ -12 \\ -14 \\ -21 \end{bmatrix}$$

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Comment. Note that, because of the way we discretize, it matters that there is a well-defined temperature at the boundary vertex $(1, 1)$. For the other vertices, we don't need a well-defined temperature (and so it is not a problem that it is unclear what the temperature should be at $(0, 0)$ or $(0, 2)$ where it jumps from 5 to 7).

Problem 5. Find all eigenfunctions and eigenvalues of

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(3) = 0.$$

Solution. We distinguish three cases:

$\lambda < 0$. The characteristic roots are $\pm r = \pm\sqrt{-\lambda}$ and the general solution to the DE is $y(x) = Ae^{rx} + Be^{-rx}$. Then $y'(0) = Ar - Br = 0$ implies $B = A$, so that $y(3) = A(e^{3r} + e^{-3r})$. Since $e^{3r} + e^{-3r} > 0$, we see that $y(3) = 0$ only if $A = 0$. So there is no solution for $\lambda < 0$.

$\lambda = 0$. The general solution to the DE is $y(x) = A + Bx$. Then $y'(0) = 0$ implies $B = 0$, and it follows from $y(3) = A = 0$ that $\lambda = 0$ is not an eigenvalue.

$\lambda > 0$. The characteristic roots are $\pm i\sqrt{\lambda}$. So, with $r = \sqrt{\lambda}$, the general solution is $y(x) = A\cos(rx) + B\sin(rx)$. $y'(0) = Br = 0$ implies $B = 0$. Then $y(3) = A\cos(3r) = 0$. Note that $\cos(3r) = 0$ is true if and only if $3r = \frac{\pi}{2} + n\pi = \frac{(2n+1)\pi}{2}$ for some integer n. Since $r > 0$, we have $n \geqslant 0$. Correspondingly, $\lambda = r^2 = \left(\frac{(2n+1)\pi}{6}\right)^2$ and $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

In summary, we have that the eigenvalues are $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$, with $n = 0, 1, 2, \ldots$ with corresponding eigenfunctions $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

Problem 6. Find the solution $u(x, t)$, for $0 < x < 3$ and $t \geqslant 0$, to the heat conduction problem

$$2u_t = u_{xx}, \quad u_x(0, t) = 0, \; u(3, t) = 0, \quad u(x, 0) = 2\cos\left(\frac{\pi x}{2}\right) + 7\cos\left(\frac{3\pi x}{2}\right).$$

Derive your solution using separation of variables (at some step you may refer to the EVP above).

Solution.

- Using separation of variables, we look for solutions $u(x, t) = X(x)T(t)$. Plugging into the PDE, we get $2X(x)T'(t) = X''(x)T(t)$, and so $\frac{X''(x)}{X(x)} = \frac{2T'(t)}{T(t)} = \text{const}$. We thus have $X'' - \text{const}\, X = 0$ and $2T' - \text{const}\, T = 0$.
- $u_x(0, t) = X'(0)T(t) = 0$ implies $X'(0) = 0$. Likewise, $u(3, t) = X(3)T(t) = 0$ implies $X(3) = 0$.
- So X solves $X'' + \lambda X = 0$ (we choose $\lambda = -\text{const}$), $X'(0) = 0$, $X(3) = 0$. We solved this EVP above and found that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$ corresponding to $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$, $n = 0, 1, 2, 3\ldots$.
- T solves $2T' + \lambda T = 0$, and hence, up to multiples, $T(t) = e^{-\frac{1}{2}\lambda t} = e^{-\frac{1}{2}\left(\frac{(2n+1)\pi}{6}\right)^2 t}$.
- Taken together, we have the solutions $u_n(x, t) = e^{-\frac{1}{2}\left(\frac{(2n+1)\pi}{6}\right)^2 t}\cos\left(\frac{(2n+1)\pi}{6}x\right)$ solving $2u_t = u_{xx}$ and $u_x(0, t) = u(3, t) = 0$.

Note that $u_n(x, 0) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$. In particular, our heat conduction problem is solved by

$$u(x, t) = 2u_1(x, t) + 7u_4(x, t) = 2e^{-\frac{1}{8}\pi^2 t}\cos\left(\frac{\pi x}{2}\right) + 7e^{-\frac{9}{8}\pi^2 t}\cos\left(\frac{3\pi x}{2}\right).$$

Comment. It is not obvious that *every* initial temperature distribution $f(x)$ can be written as an (infinite) superposition of the $u_n(x, 0)$. However, such "eigenfunction expansions" are always possible (this extends what we know about ordinary Fourier series).