Systems of recurrence equations

Example 55. Write the (second-order) RE $a_{n+2} = a_{n+1} + 2a_n$, with $a_0 = 1$, $a_1 = 8$, as a system of (first-order) recurrences.

Solution. Write $b_n = a_{n+1}$.

Then, $a_{n+2} = a_{n+1} + 2a_n$ translates into the first-order system $\begin{cases} a_{n+1} = b_n \\ b_{n+1} = 2a_n + b_n \end{cases}$.

Let $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$. Then, in matrix form, the RE is $\boldsymbol{a}_{n+1} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{a}_n$, with $\boldsymbol{a}_0 = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$.

Comment. Consequently, $\boldsymbol{a}_n = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^n \boldsymbol{a}_0 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 8 \end{bmatrix}$. Solving (systems of) REs is equivalent to computing powers of matrices!

Example 56. Determine the general solution to $\boldsymbol{a}_{n+1} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{a}_n$.

Solution. In the previous example, we obtained this system from the RE $a_{n+2} = a_{n+1} + 2a_n$, which we know (do it!) has solutions $a_n = 2^n$ and $a_n = (-1)^n$ (which combine to the general solution $a_n = C_1 \cdot 2^n + C_2 \cdot (-1)^n$).

Correspondingly, $\boldsymbol{a}_{n+1} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{a}_n$ has solutions $\boldsymbol{a}_n = \begin{bmatrix} 2^n \\ 2^{n+1} \end{bmatrix}$ and $\boldsymbol{a}_n = \begin{bmatrix} (-1)^n \\ (-1)^{n+1} \end{bmatrix}$.

These combine to the general solution $\boldsymbol{a}_n = C_1 \begin{bmatrix} 2^n \\ 2^{n+1} \end{bmatrix} + C_2 \begin{bmatrix} (-1)^n \\ (-1)^{n+1} \end{bmatrix} = \begin{bmatrix} 2^n & (-1)^n \\ 2^{n+1} & (-1)^{n+1} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.

We call $\Phi_n = \begin{bmatrix} 2^n & (-1)^n \\ 2^{n+1} & (-1)^{n+1} \end{bmatrix}$ a **fundamental matrix (solution)**. The general solution is $\Phi_n \boldsymbol{c}$ with $\boldsymbol{c} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.

Observations.

(a) The columns of Φ_n are (independent) solutions of the system.

(b) Φ_n solves the RE itself: $\Phi_{n+1} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \Phi_n$.

[Spell this out in this example! That Φ_n solves the RE follows from the definition of matrix multiplication.]

(c) It follows that $\Phi_n = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^n \Phi_0$. Equivalently, $\Phi_n \Phi_0^{-1} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^n$. (See next example!)

Matrix powers M^n can be computed by diagonalizing the matrix M (if you have taken linear algebra classes, you might have seen this).

We just saw that being able to compute matrix powers is equivalent to solving systems of recurrences. In the next example, we use this connection to compute some matrix powers.

(a way to compute powers of a matrix M)

Compute a fundamental matrix solution Φ_n of $\boldsymbol{a}_{n+1} = M \boldsymbol{a}_n$.

Then $M^n = \Phi_n \Phi_0^{-1}$.

Example 57. Compute M^n for $M = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

Solution. We already observed that $\Phi_n = \begin{bmatrix} 2^n & (-1)^n \\ 2^{n+1} & (-1)^{n+1} \end{bmatrix}$ is a fundamental matrix solution Φ_n of $\boldsymbol{a}_{n+1} = M \boldsymbol{a}_n$.

We have $\Phi_0^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$. Hence,

$$M^n = \Phi_n \Phi_0^{-1} = \begin{bmatrix} 2^n & (-1)^n \\ 2^{n+1} & (-1)^{n+1} \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2^n + 2(-1)^n & 2^n - (-1)^n \\ 2^{n+1} + 2(-1)^{n+1} & 2^{n+1} - (-1)^{n+1} \end{bmatrix}.$$

Note. M^n is a fundamental matrix solution of $\boldsymbol{a}_{n+1} = M \boldsymbol{a}_n$ itself.

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Example 58. Let $M = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(b) Determine a fundamental matrix solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(c) Compute M^n.

Solution.

(a) Let us look for solutions of the form $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$ (where $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$). Note that $\boldsymbol{a}_{n+1} = \boldsymbol{v}\lambda^{n+1} = \lambda\boldsymbol{a}_n$.
Plugging into $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ we find $\boldsymbol{v}\lambda^{n+1} = M\boldsymbol{v}\lambda^n$.
Cancelling λ^n (just a number!), this simplifies to $\lambda\boldsymbol{v} = M\boldsymbol{v}$.
In other words, $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$ is a solution if and only if $\boldsymbol{v}$ is a λ-eigenvector of M.
We computed earlier that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$.
Hence, the general solution is $C_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}3^n + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}(-2)^n$.

(b) The corresponding fundamental matrix solution is $\Phi_n = \begin{bmatrix} 2\cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix}$.
[Note that our general solution is precisely $\Phi_n\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.]

(c) Note that $\Phi_0 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, so that $\Phi_0^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. It follows that

$$M^n = \Phi_n\Phi_0^{-1} = \begin{bmatrix} 2\cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 3^n - (-2)^n & -2\cdot 3^n + 2(-2)^n \\ 3^n - (-2)^n & -3^n + 2(-2)^n \end{bmatrix}.$$

To solve $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, determine the eigenvectors of M.

- Each λ-eigenvector $\boldsymbol{v}$ provides a solution: $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$
- If there are enough eigenvectors, these combine to the general solution.

Comment. If there are not enough eigenvectors, then we know what to do as well (at least in principle): instead of looking only for solutions of the type $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$, we also need to look for solutions of the type $\boldsymbol{a}_n = (\boldsymbol{v}n + \boldsymbol{w})\lambda^n$. Note that this can only happen if an eigenvalue is a repeated root of the characteristic polynomial.

Example 59. Write $a_{n+3} - 4a_{n+2} + a_{n+1} + 6a_n = 0$ as a system of (first-order) recurrences.

Solution. Write $b_n = a_{n+1}$ and $c_n = a_{n+2}$.

Then, $a_{n+3} - 4a_{n+2} + a_{n+1} + 6a_n = 0$ translates into the first-order system $\begin{cases} a_{n+1} = b_n \\ b_{n+1} = c_n \\ c_{n+1} = -6a_n - b_n + 4c_n \end{cases}$.

Let $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}$. Then, in matrix form, the RE is $\boldsymbol{a}_{n+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix} \boldsymbol{a}_n$.

Review.

- Consequently, $\boldsymbol{a}_n = M^n \boldsymbol{a}_0$, where M is the matrix above.
- In general, we can solve $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ by finding the eigenvectors of M:
  An λ-eigenvector $\boldsymbol{v}$ provides the solution $\boldsymbol{a}_n = \boldsymbol{v}\lambda^n$.
- Here, because we started with a single (third-order) equation, we can avoid computing eigenvectors:
  $a_n = C_1 \cdot 3^n + C_2 \cdot 2^n + C_3 \cdot (-1)^n$ is the general solution to the initial RE. (Why?! Do it!)

  Correspondingly, a fundamental matrix solution of the system is $\Phi_n = \begin{bmatrix} 3^n & 2^n & (-1)^n \\ 3 \cdot 3^n & 2 \cdot 2^n & -(-1)^n \\ 9 \cdot 3^n & 4 \cdot 2^n & (-1)^n \end{bmatrix}$.

  Note. This tells us that $\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$ is a 3-eigenvector, $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ a 2-eigenvector, and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ a -1-eigenvector of M.
- Since $\Phi_{n+1} = M\Phi_n$, we have $\Phi_n = M^n \Phi_0$ so that $M^n = \Phi_n \Phi_0^{-1}$. This allows us to compute that:
  $$M^n = \frac{1}{12} \begin{bmatrix} -6 \cdot 3^n + 12 \cdot 2^n + 6(-1)^n & -3 \cdot 3^n + 8 \cdot 2^n - 5(-1)^n & 3 \cdot 3^n - 4 \cdot 2^n + (-1)^n \\ -18 \cdot 3^n + 24 \cdot 2^n - 6(-1)^n & \ldots & \ldots \\ -54 \cdot 3^n + 48 \cdot 2^n + 6(-1)^n & \ldots & \ldots \end{bmatrix}$$

(systems of REs) The unique solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$, $\boldsymbol{a}_0 = \boldsymbol{c}$ is $\boldsymbol{a}_n = M^n \boldsymbol{c}$.

Note that M^n is the fundamental matrix solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ with $\boldsymbol{a}_0 = I$ (the identity matrix).

Systems of differential equations

Example 60. Write the (second-order) differential equation $y'' = 2y' + y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$ and $y_2 = y'$. Then $y'' = 2y' + y$ becomes $y_2' = 2y_2 + y_1$.

Therefore, $y'' = 2y' + y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_1 + 2y_2 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \boldsymbol{y}$.

Comment. Hence, we care about systems of differential equations, even if we work with just one function.

Example 61. Write the (third-order) differential equation $y''' = 3y'' - 2y' + y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' = 3y'' - 2y' + y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_1 - 2y_2 + 3y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix} \boldsymbol{y}$.

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Example 62. Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 2y_1' - 3y_2' + 7y_2 \\ y_2'' &= 4y_1' + y_2' - 5y_1 \end{aligned} \qquad y_1(0) = 2,\ y_1'(0) = 3,\ y_2(0) = -1,\ y_2'(0) = 1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

(systems of DEs) The unique solution to $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{c}$ is $\boldsymbol{y}(x) = e^{Mx}\boldsymbol{c}$.

Here, e^{Mx} is the fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = I$ (with I the identity matrix).

Important. We are defining the **matrix exponential** e^{Mx} as the solution to an IVP. This is equivalent to how one can define the ordinary exponential e^x as the solution to $y' = y$, $y(0) = 1$.

[In a little bit, we will also discuss how to think about the matrix exponential e^{Mx} using power series.]

(a way to compute the matrix exponential e^{Mx})

Compute a fundamental matrix solution $\Phi(x)$ of $\boldsymbol{y}' = M\boldsymbol{y}$.

Then $e^{Mx} = \Phi(x)\Phi(0)^{-1}$.

Compare this to our method of computing matrix powers M^n.

Proof. If $\Phi(x)$ is a fundamental matrix solution, then so is $\Psi(x) = \Phi(x)C$ for any constant matrix C. (Why?!)

Therefore, $\Psi(x) = \Phi(x)\Phi(0)^{-1}$ is a fundamental matrix solution with $\Psi(0) = \Phi(0)\Phi(0)^{-1} = I$.

But e^{Mx} is defined to be the unique such solution, so that $\Psi(x) = e^{Mx}$.

Observe how the next example proceeds along the same lines as Example 58.

Example 63. Let $M = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

Solution.

(a) Let us look for solutions of the form $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$ (where $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$). Note that $\boldsymbol{y}' = \lambda\boldsymbol{v}e^{\lambda x} = \lambda\boldsymbol{y}$.

Plugging into $\boldsymbol{y}' = M\boldsymbol{y}$ we find $\lambda\boldsymbol{y} = M\boldsymbol{y}$.

In other words, $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$ is a solution if and only if $\boldsymbol{v}$ is a λ-eigenvector of M.

We computed earlier that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$.

Hence, the general solution is $C_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{3x} + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2x}$.

(b) The corresponding fundamental matrix solution is $\Phi = \begin{bmatrix} 2 \cdot e^{3x} & e^{-2x} \\ e^{3x} & e^{-2x} \end{bmatrix}$.

[Note that our general solution is precisely $\Phi\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.]

(c) Note that $\Phi(0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, so that $\Phi(0)^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} 2 \cdot e^{3x} & e^{-2x} \\ e^{3x} & e^{-2x} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot e^{3x} - e^{-2x} & -2 \cdot e^{3x} + 2e^{-2x} \\ e^{3x} - e^{-2x} & -e^{3x} + 2e^{-2x} \end{bmatrix}.$$

Example 64. Let $p(D) = D^m + c_{m-1}D^{m-1} + \ldots + c_1 D + c_0$. Write the DE $p(D)y = 0$ as a system of (first-order) differential equations.

Solution. Write $y_k = D^k y$ for $k = 0, 1, \ldots, m-1$.

Then, $p(D)y = 0$ translates into the first-order system $\begin{cases} y_0' = y_1 \\ y_1' = y_2 \\ \vdots \\ y_{m-1}' = -c_{m-1}y_{m-1} - \ldots - c_1 y_1 - c_0 y_0 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & \cdots & \cdots & -c_{m-1} \end{bmatrix} \boldsymbol{y}$.

Comment. This is called the **companion matrix** of the polynomial $p(D)$. Can you see why the characteristic polynomial of the matrix must be (up to possibly a sign) equal to $p(D)$?

As expected, this works exactly the same way for recurrence equations:

Example 65. (extra) Let $p(N) = N^m + c_{m-1}N^{m-1} + \ldots + c_1 N + c_0$. Write the RE $p(N)a_n = 0$ as a system of (first-order) recurrences.

Solution. Write $a_n^{(k)} = N^k a_n = a_{n+k}$ for $k = 0, 1, \ldots, m-1$.

Then, $p(N)a_n = 0$ translates into the first-order system $\begin{cases} a_{n+1}^{(0)} = a_n^{(1)} \\ a_{n+1}^{(1)} = a_n^{(2)} \\ \vdots \\ a_{n+1}^{(m-1)} = -c_{m-1}a_n^{(m-1)} - \ldots - c_1 a_n^{(1)} - c_0 a_n^{(0)} \end{cases}$.

Let $\boldsymbol{a}_n = \begin{bmatrix} a_n^{(0)} \\ a_n^{(1)} \\ \vdots \\ a_n^{(m-1)} \end{bmatrix}$. Then, in matrix form, the RE is: $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ with $M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & \cdots & \cdots & -c_{m-1} \end{bmatrix}$

To solve $\boldsymbol{y}' = M\boldsymbol{y}$, determine the eigenvectors of M.

- Each λ-eigenvector $\boldsymbol{v}$ provides a solution: $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$
- If there are enough eigenvectors, these combine to the general solution.

Comment. If there are not enough eigenvectors, then we know what to do as well (at least in principle): instead of looking only for solutions of the type $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$, we also need to look for solutions of the type $\boldsymbol{y}(x) = (\boldsymbol{v}x + \boldsymbol{w})e^{\lambda x}$. Note that this can only happen if an eigenvalue is a repeated root of the characteristic polynomial.

Example 66. Let $M = \begin{bmatrix} -1 & 6 \\ -1 & 4 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

(d) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

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Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M-\lambda I) = \det\left(\begin{bmatrix} -1-\lambda & 6 \\ -1 & 4-\lambda \end{bmatrix}\right) = (-1-\lambda)(4-\lambda)+6 = \lambda^2-3\lambda+2 = (\lambda-1)(\lambda-2)$

Hence, the eigenvalues are $\lambda=1$ and $\lambda=2$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda=1$, we need to solve $\begin{bmatrix} -2 & 6 \\ -1 & 3 \end{bmatrix}\boldsymbol{v}=\mathbf{0}$.
  Hence, $\boldsymbol{v}=\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda=1$.
- To find an eigenvector $\boldsymbol{v}$ for $\lambda=2$, we need to solve $\begin{bmatrix} -3 & 6 \\ -1 & 2 \end{bmatrix}\boldsymbol{v}=\mathbf{0}$.
  Hence, $\boldsymbol{v}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda=2$.

Hence, the general solution is $C_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^x + C_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{2x}$.

(b) The corresponding fundamental matrix solution is $\Phi = \begin{bmatrix} 3e^x & 2e^{2x} \\ e^x & e^{2x} \end{bmatrix}$.

(c) Note that $\Phi(0)=\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, so that $\Phi(0)^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} 3e^x & 2e^{2x} \\ e^x & e^{2x} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3e^x-2e^{2x} & -6e^x+6e^{2x} \\ e^x-e^{2x} & -2e^x+3e^{2x} \end{bmatrix}.$$

(d) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3e^x-2e^{2x} & -6e^x+6e^{2x} \\ e^x-e^{2x} & -2e^x+3e^{2x} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3e^x+4e^{2x} \\ -e^x+2e^{2x} \end{bmatrix}$.

Note. If we hadn't already computed e^{Mx}, we would use the general solution and solve for the appropriate values of C_1 and C_2. Do it that way as well!

Theorem 67. Let M be $n\times n$. Then the **matrix exponential** satisfies

$$e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \ldots$$

Proof. Define $\Phi(x) = I + Mx + \frac{1}{2!}M^2x^2 + \frac{1}{3!}M^3x^3 + \ldots$

$$\begin{aligned} \Phi'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[I + Mx + \frac{1}{2!}M^2x^2 + \frac{1}{3!}M^3x^3 + \ldots\right] \\ &= 0 + M + M^2x + \frac{1}{2!}M^3x^2 + \ldots = M\Phi(x). \end{aligned}$$

Clearly, $\Phi(0)=I$. Therefore, $\Phi(x)$ is the fundamental matrix solution to $\boldsymbol{y}'=M\boldsymbol{y}$, $\boldsymbol{y}(0)=I$.
But that's precisely how we defined e^{Mx} earlier. It follows that $\Phi(x)=e^{Mx}$. □

(exponential function) e^x is the unique solution to $y'=y$, $y(0)=1$.
From here, it follows that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$.

The latter is the Taylor series for e^x at $x=0$ that we have seen in Calculus II.
Important note. We can actually construct this infinite sum directly from $y'=y$ and $y(0)=1$.
Indeed, observe how each term, when differentiated, produces the term before it. For instance, $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3}{3!} = \frac{x^2}{2!}$.

Example 68. If $A=\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $A^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix}$.

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Example 69. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2^2 & 0 \\ 0 & 5^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$.

Clearly, this works to obtain e^D for any diagonal matrix D.

In particular, for $Ax = \begin{bmatrix} 2x & 0 \\ 0 & 5x \end{bmatrix}$, $e^{Ax} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2x & 0 \\ 0 & 5x \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (2x)^2 & 0 \\ 0 & (5x)^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^{2x} & 0 \\ 0 & e^{5x} \end{bmatrix}$.

Example 70. Let $M = \begin{bmatrix} 8 & 4 \\ -1 & 4 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

(d) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M - \lambda I) = \det\left(\begin{bmatrix} 8-\lambda & 4 \\ -1 & 4-\lambda \end{bmatrix}\right) = (8-\lambda)(4-\lambda) + 4 = \lambda^2 - 12\lambda + 36 = (\lambda - 6)(\lambda - 6)$

Hence, the eigenvalues are $\lambda = 6, 6$ (meaning that 6 has multiplicity 2).

- To find eigenvectors $\boldsymbol{v}$ for $\lambda = 6$, we need to solve $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$.
  Hence, $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 6$. There is no independent second eigenvector.

- We therefore search for a solution of the form $\boldsymbol{y}(x) = (\boldsymbol{v}x + \boldsymbol{w})e^{\lambda x}$ with $\lambda = 6$.
  $\boldsymbol{y}'(x) = (\lambda\boldsymbol{v}x + \lambda\boldsymbol{w} + \boldsymbol{v})e^{\lambda x} \overset{!}{=} M\boldsymbol{y} = (M\boldsymbol{v}x + M\boldsymbol{w})e^{\lambda x}$
  Equating coefficients of x, we need $\lambda\boldsymbol{v} = M\boldsymbol{v}$ and $\lambda\boldsymbol{w} + \boldsymbol{v} = M\boldsymbol{w}$.
  Hence, $\boldsymbol{v}$ must be an eigenvector (which we already computed); we choose $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
  [Note that any multiple of $\boldsymbol{y}(x)$ will be another solution, so it doesn't matter which multiple of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ we choose.]
  $\lambda\boldsymbol{w} + \boldsymbol{v} = M\boldsymbol{w}$ or $(M - \lambda)\boldsymbol{w} = \boldsymbol{v}$ then becomes $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\boldsymbol{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
  One solution is $\boldsymbol{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. [We only need one.]

Hence, the general solution is $C_1\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{6x} + C_2\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}x + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)e^{6x}$.

(b) The corresponding fundamental matrix solution is $\Phi = \begin{bmatrix} -2e^{6x} & -(2x+1)e^{6x} \\ e^{6x} & x\,e^{6x} \end{bmatrix}$.

(c) Note that $\Phi(0) = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$, so that $\Phi(0)^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} -2e^{6x} & -(2x+1)e^{6x} \\ e^{6x} & x\,e^{6x} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} (2x+1)e^{6x} & 4x\,e^{6x} \\ -x\,e^{6x} & -(2x-1)\,e^{6x} \end{bmatrix}.$$

(d) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (2x+1)e^{6x} & 4x\,e^{6x} \\ -x\,e^{6x} & -(2x-1)\,e^{6x} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (2x+1)e^{6x} \\ -x\,e^{6x} \end{bmatrix}$.

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