Solving the heat equation

EG Solve:
$$u_t = k u_{xx} \quad \text{PDE}$$
$$u(0,t) = u(L,t) = 0 \quad \text{BC}$$
$$u(x,0) = f(x) \quad \text{IC}$$

- look for $u(x,t) = X(x)\,T(t)$ separation of variables
- PDE $X(x)\,T'(t) = k\,X''(x)\,T(t)$
$$\frac{X''(x)}{X(x)} = \frac{1}{k}\,\frac{T'(t)}{T(t)} = \text{constant} =: -\lambda$$
$$\Longrightarrow \quad X'' + \lambda X = 0 \,, \qquad T' + \lambda k T = 0$$
- BC
$$u(0,t) = X(0)\,T(t) = 0 \quad \rightsquigarrow \quad X(0) = 0$$
$$u(L,t) = X(L)\,T(t) = 0 \quad \rightsquigarrow \quad X(L) = 0$$
- $X'' + \lambda X = 0 \,, \quad X(0) = 0, \; X(L) = 0$

eigenvalue problem $\quad \lambda = \left(\frac{\pi n}{L}\right)^2$, $n = 1, 2, 3, \ldots$ eigenvalues $\qquad X(x) = \sin\left(\frac{\pi n}{L}x\right)$ eigenfunctions

- $T' + \lambda k T = 0$
$$T(t) = e^{-\lambda k t} = e^{-\left(\frac{\pi n}{L}\right)^2 k t}$$
- PDE + BC solved by $\quad u_n(x,t) = e^{-\left(\frac{\pi n}{L}\right)^2 k t} \sin\left(\frac{\pi n}{L}x\right)$
- IC $\quad u(x,0) = f(x) \qquad u_n(x,0) = \sin\left(\frac{\pi n}{L}x\right)$
$$= \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{L}x\right) \quad \text{Fourier sine series of } f(x)$$

$$\Longrightarrow \quad u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{\pi n}{L}\right)^2 k t} \sin\left(\frac{\pi n}{L}x\right)$$