Homogeneous LDEs with constant coefficients

part 3

characteristic polynomial

THM $p(D)y = 0$

- If r is a root of $p(D)$ of multiplicity k, then k solutions are:
  $$x^j e^{rx} \qquad j = 0, 1, \dots, k-1$$
- Combining these, for all r, gives the general solution.

Pf $p(D) = q(D)(D-r)^k$

need: k solutions to $(D-r)^k y = 0$

idea: try $y(x) = c(x)e^{rx}$ like variation of constants

$$(D-r)\,c(x)e^{rx} = c'(x)e^{rx} + \cancel{c(x)re^{rx} - rc(x)e^{rx}}$$
$$= c'(x)e^{rx}$$
$$(D-r)^2 c(x)e^{rx} = (D-r)\,c'(x)e^{rx} = c''(x)e^{rx}$$
$$\vdots$$
$$(D-r)^k c(x)e^{rx} = c^{(k)}(x)\,e^{rx}$$

$$c^{(k)}(x)e^{rx} = 0$$
$\Leftrightarrow$ $c^{(k)}(x) = 0$
$\Leftrightarrow$ $c(x)$ is polynomial of degree less than k
$c_0 + c_1 x + \dots + c_{k-1}x^{k-1}$

EG $y''' - 3y' + 2y = 0$

$p(D)\,y = 0$ $\qquad$ $p(D) = D^3 - 3D + 2$
$$= (D-1)^2(D+2)$$
roots: $1, 1, -2$
$$e^x,\ xe^x,\ e^{-2x}$$

general solution:
$$(c_1 + c_2 x)e^x + c_3 e^{-2x}$$