Power series of familiar functions

(Unless we specify otherwise, power series are meant to be about $x=0$.)

Example 98. The **hyperbolic cosine** $\cosh(x)$ is defined to be the even part of e^x. In other words, $\cosh(x)=\frac{1}{2}(e^x+e^{-x})$. Determine its power series.

Solution. It follows from $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ that $\cosh(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$.

Comment. Note that $\cosh(ix)=\cos(x)$ (because $\cos(x)=\frac{1}{2}(e^{ix}+e^{-ix})$).

Comment. The hyperbolic sine $\sinh(x)$ is similarly defined to be the odd part of e^x.

Example 99. (geometric series) Determine $y(x)=\sum_{n=0}^{\infty}x^n$.

Solution. Note that $xy=y-1$. Hence, $y=\frac{1}{1-x}$.

Comment. The radius of convergence of this series is 1. This is easy to see directly. But note that it also follows from Theorem 88 since $y(x)$ solves the "differential" (inhomogeneous) equation $(1-x)y=1$, for which the only singular point is $x=1$.

Example 100. Determine a power series for $\frac{1}{1+x^2}$.

Solution. Replace x with $-x^2$ in $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ to get $\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}$.

Example 101. (extra) Determine a power series for $\ln(x)$ around $x=1$.

Solution. This is equivalent to finding a power series for $\ln(x+1)$ around $x=0$ (see the final step).

Observe that $\ln(x+1)=\int\frac{\mathrm{d}x}{1+x}$ and that $\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n$.

Integrating, $\ln(x+1)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}+C$. Since $\ln(1)=0$, we conclude that $C=0$.

Finally, $\ln(x+1)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$ is equivalent to $\ln(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(x-1)^{n+1}$.

Comment. Choosing $x=2$ in $\ln(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(x-1)^{n+1}$ results in $\ln(2)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$.
The latter is the alternating harmonic sum.
Can you see from here why the harmonic sum $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ diverges?

Example 102. Determine a power series for $\arctan(x)$.

Solution. Recall that $\arctan(x)=\int\frac{\mathrm{d}x}{1+x^2}$. Hence, we need to integrate $\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}$.

It follows that $\arctan(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}+C$. Since $\arctan(0)=0$, we conclude that $C=0$.

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Example 103. (error function) Determine a power series for $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}t$.

Solution. It follows from $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ that $e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$.

Integrating, we obtain $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}t = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$.

Example 104. Determine the first several terms (up to x^5) in the power series of $\tan(x)$.

Solution. Observe that $y(x) = \tan(x)$ is the unique solution to the IVP $y' = 1 + y^2$, $y(0) = 0$.

We can therefore proceed to determine the first few power series coefficients as we did earlier.

That is, we plug $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$ into the DE. Note that $y(0) = 0$ means $a_0 = 0$.

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots$

$1 + y^2 = 1 + (a_1 x + a_2 x^2 + a_3 x^3 + \ldots)^2 = 1 + a_1^2 x^2 + (2a_1 a_2)x^3 + (2a_1 a_3 + a_2^2)x^4 + \ldots$

Comparing coefficients, we find: $a_1 = 1$, $2a_2 = 0$, $3a_3 = a_1^2$, $4a_4 = 2a_1 a_2$, $5a_5 = 2a_1 a_3 + a_2^2$.

Solving for $a_2, a_3, \ldots$, we conclude that $\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots$

Comment. The fact that $\tan(x)$ is an odd function translates into $a_n = 0$ when n is even. If we had realized that at the beginning, our computation would have been simplified.

Advanced comment. The full power series is $\tan(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} x^{2n-1}$.

Here, the numbers B_{2n} are (rather mysterious) rational numbers known as **Bernoulli numbers**.

The radius of convergence is $\pi/2$. Note that this is not at all obvious from the DE $y' = 1 + y^2$. This illustrates the fact that nonlinear DEs are much more complicated than linear ones. (There's no analog of Theorem 88.)

Fourier series

The following amazing fact is saying that any 2π-periodic function can be written as a sum of cosines and sines.

Advertisement. In Linear Algebra II, we will see the following natural way to look at Fourier series: the functions $1, \cos(t), \sin(t), \cos(2t), \sin(2t), \ldots$ are orthogonal to each other (for that to make sense, we need to think of functions as vectors and introduce a natural inner product). In fact, they form an orthogonal basis for the space of piecewise smooth functions. In that setting, the formulas for the coefficients a_n and b_n are nothing but the usual projection formulas for orthogonal projection onto a single vector.

Theorem 105. Every* 2π-periodic function f can be written as a **Fourier series**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)).$$

Technical detail*: f needs to be, e.g., piecewise smooth.

Also, if t is a discontinuity of f, then the Fourier series converges to the average $\frac{f(t^-) + f(t^+)}{2}$.

The **Fourier coefficients** a_n, b_n are unique and can be computed as

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\mathrm{d}t, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\mathrm{d}t.$$

Comment. Another common way to write Fourier series is $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$.

These two ways are equivalent; we can convert between them using Euler's identity $e^{int} = \cos(nt) + i\sin(nt)$.