Sketch of Lecture 13

Tue, 10/8/2019

Two more applications of systems of DEs

Example 75. (epidemiology) Let us indicate the popular SIR model for short outbreaks of diseases among a population of constant size N.

In a SIR model, the population is compartmentalized into $S(t)$ susceptible, $I(t)$ infected and $R(t)$ recovered (or resistant) individuals ($N = S(t) + I(t) + R(t)$). In the Kermack-McKendrick model, the outbreak of a disease is modeled by

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I,$$

with γ modeling the recovery rate and β the infection rate. Note that this is a non-linear system of differential equations. For more details and many variations used in epidemiology, see:

`https://en.wikipedia.org/wiki/Compartmental_models_in_epidemiology`

Comment. The following variation

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I R, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I R,$$

which assumes "infectious recovery", was recently used to predict that facebook might loose 80% of its users by 2017. It's that claim, not mathematics (or even the modeling), which attracted a lot of media attention.

`http://blogs.wsj.com/digits/2014/01/22/controversial-paper-predicts-facebook-decline/`

Example 76. (military strategy) Lanchester's equations model two opposing forces during "aimed fire" battle.

Let $x(t)$ and $y(t)$ describe the number of troops on each side. Then Lanchester (during World War I) assumed that the rates $x'(t)$ and $y'(t)$, at which soldiers are put out of action, are proportional to the number of opposing forces. That is:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\alpha y(t) \\ -\beta x(t) \end{bmatrix}, \quad \text{or, in matrix form: } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \\ -\beta & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The proportionality constants $\alpha, \beta > 0$ indicate the strength of the forces ("fighting effectiveness coefficients").

These are simple linear DEs with constant coefficients, which we have learned how to solve.

For more details, see: `https://en.wikipedia.org/wiki/Lanchester%27s_laws`

Comment. The "aimed fire" means that all combatants are engaged, as is common in modern combat with long-range weapons. This is rather different than ancient combat, where soldier's were engaging one opponent at a time.

Some special functions and the power series method

Review: power series

Definition 77. A function $y(x)$ is analytic around $x = x_0$ if it has a power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

In other words, $y(x)$ is equal to its Taylor series around $x = x_0$.

Power series are very pleasant to work with because they behave just like polynomials. For instance, we can differentiate and integrate them:

- If $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, then $y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$ (another power series!).

  Note that $y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n$. Likewise, for higher derivatives.

- $\int y(x) \mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1} + C$

Theorem 78. If $y(x)$ is analytic around $x = x_0$, then $y(x)$ is infinitely differentiable and

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{with} \quad a_n = \frac{y^{(n)}(x_0)}{n!}.$$

Caution. Analyticity is needed in this theorem; being infinitely differentiable is not enough. For instance, $y(x) = e^{-1/x^2}$ is infinitely differentiable around $x = 0$ (and everywhere else). However, $y^{(n)}(0) = 0$ for all n.

We have already seen the following example.

Example 79. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \ldots$

Once again, notice how the power series clearly has the property that $y' = y$.

It follows from here that, for instance, $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + 2x^2 + \frac{4}{3} x^3 + \ldots$

Example 80. Determine a power series for $\cos(x)$.

Solution. (via DE) $\cos(x)$ is the unique solution to the IVP $y'' = -y$, $y(0) = 1$, $y'(0) = 0$.

It follows that $\cos(x) = \sum_{n=0}^{\infty} a_n x^n$ with $a_n = \frac{y^{(n)}(0)}{n!}$. The DE implies that $y^{(2n)}(x) = (-1)^n y(x)$ and $y^{(2n+1)} = (-1)^n y'(x)$ so that $y^{(2n)}(0) = (-1)^n$ and $y^{(2n+1)}(0) = 0$. Consequently, $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$.

Solution. (via Euler's formula) Recall that $e^{ix} = \cos(x) + i \sin(x)$. Since

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{m=0}^{\infty} \frac{(ix)^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(ix)^{2m+1}}{(2m+1)!} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!},$$

we conclude that $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ and $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$.

Example 81. (Airy equation, to be cont'd) Let $y(x)$ be the unique solution to the IVP $y'' = xy$, $y(0) = a$, $y'(0) = b$. Determine the first several terms (up to x^6) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y''(0) = 0 \cdot y(0) = 0$.

Differentiating both sides of the DE, we obtain $y''' = y + xy'$ so that $y'''(0) = y(0) + 0 \cdot y'(0) = a$.

Likewise, $y^{(4)} = 2y' + xy''$ shows $y^{(4)}(0) = 2y'(0) = 2b$.

Continuing, $y^{(5)} = 3y'' + xy'''$ so that $y^{(5)}(0) = 3y''(0) = 0$.

Continuing, $y^{(6)} = 4y''' + xy^{(4)}$ so that $y^{(6)}(0) = 4y'''(0) = 4a$.

Hence, $y(x) = a + bx + \frac{1}{2} y''(0) x^2 + \frac{1}{6} y'''(0) x^3 + \frac{1}{24} y^{(4)}(0) x^4 + \frac{1}{120} y^{(5)}(0) x^5 + \frac{1}{720} y^{(6)}(0) x^6 + \ldots$
$= a + bx + \frac{a}{6} x^3 + \frac{b}{12} x^4 + \frac{a}{180} x^6 + \ldots$

Comment. Do you see the general pattern? We will revisit this example soon.

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