Crash course: Eigenvalues and eigenvectors

If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ (and $\boldsymbol{x} \neq \boldsymbol{0}$), then $\boldsymbol{x}$ is an **eigenvector** of A with **eigenvalue** λ (just a number).

Note that for the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to make sense, A needs to be a square matrix (i.e. $n \times n$).

Key observation:

$$\begin{aligned} & A\boldsymbol{x} = \lambda\boldsymbol{x} \\ \iff \; & A\boldsymbol{x} - \lambda\boldsymbol{x} = \boldsymbol{0} \\ \iff \; & (A - \lambda I)\boldsymbol{x} = \boldsymbol{0} \end{aligned}$$

This homogeneous system has a nontrivial solution $\boldsymbol{x}$ if and only if $\det(A - \lambda I) = 0$.

To find eigenvectors and eigenvalues of A:

(a) First, find the eigenvalues λ by solving $\det(A - \lambda I) = 0$.

$\det(A - \lambda I)$ is a polynomial in λ, called the **characteristic polynomial** of A.

(b) Then, for each eigenvalue λ, find corresponding eigenvectors by solving $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$.

Example 51. Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

Solution. The characteristic polynomial is:

$\det(A - \lambda I) = \det\left(\begin{bmatrix} 8-\lambda & -10 \\ 5 & -7-\lambda \end{bmatrix}\right) = (8-\lambda)(-7-\lambda) + 50 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$

Hence, the eigenvalues are $\lambda = 3$ and $\lambda = -2$.

- To find an eigenvector for $\lambda = 3$, we need to solve $\begin{bmatrix} 5 & -10 \\ 5 & -10 \end{bmatrix}\boldsymbol{x} = \boldsymbol{0}$.
  Hence, $\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$.
- To find an eigenvector for $\lambda = -2$, we need to solve $\begin{bmatrix} 10 & -10 \\ 5 & -5 \end{bmatrix}\boldsymbol{x} = \boldsymbol{0}$.
  Hence, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$.

Check! $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

On the other hand, a random other vector like $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an eigenvector: $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ -9 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Example 52. (homework) Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -6 \\ 1 & -4 \end{bmatrix}$.

Solution. (final answer only) $\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$, and $\boldsymbol{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -1$.

Back to linear recurrences

Example 53. (review) Consider the sequence a_n defined by $a_{n+2} = a_{n+1} + 2a_n$ and $a_0 = 1$, $a_1 = 8$.

(a) Determine the first few terms of the sequence.

(b) Find a Binet-like formula for a_n.

(c) Determine $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $a_2 = 10$, $a_3 = 26$

(b) The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 2$ has roots $2, -1$.

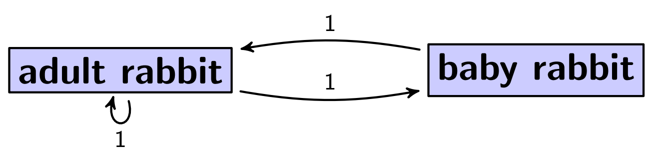
Hence, $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$ and we only need to figure out the two unknowns α_1, α_2. We can do that using the two initial conditions: $a_0 = \alpha_1 + \alpha_2 = 1$, $a_1 = 2\alpha_1 - \alpha_2 = 8$.

Solving, we find $\alpha_1 = 3$ and $\alpha_2 = -2$ so that, in conclusion, $a_n = 3 \cdot 2^n - 2(-1)^n$.

(c) It follows from the Binet-like formula that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 2$.

Example 54. We model rabbit reproduction as follows.

Each month, every pair of adult rabbits produces one pair of baby rabbit as offspring. Meanwhile, it takes baby rabbits one month to mature to adults.



Comment. In this simplified model, rabbits always come in male/female pairs and no rabbits die. Though these feautures might make it sound fairly useless, the model may have some merit when describing populations under ideal conditions (unlimited resources) and over short time (no deaths).

Historical comment. The question how many rabbits there are after one year, when starting out with a pair of baby rabbits is famously included in the 1202 textbook of the Italian mathematician Leonardo of Pisa, known as Fibonacci.

If we start with one baby rabbit pair, how many adult rabbits are there after n months?

Solution. Let a_n be the number of baby rabbit pairs after n months. Likewise, b_n is the number of adult rabbit pairs. The transition from one month to the next is given by $a_{n+1} = b_n$ and $b_{n+1} = a_n + b_n$. Using $a_n = b_{n-1}$ (that's an equivalent version of the first equation) in the second equation, we obtain $b_{n+1} = b_n + b_{n-1}$.

The initial conditions are $b_0 = 0$ and $b_1 = 0$.

It follows that the number b_n of adult rabbits are precisely the Fibonacci numbers F_n.

Comment. Note that the transition from one month to the next is described by in matrix-vector form as

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} b_n \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}.$$

Writing $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$, this becomes $\boldsymbol{a}_{n+1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \boldsymbol{a}_n$.

Consequently, $\boldsymbol{a}_n = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \boldsymbol{a}_0 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.