Final: practice

MATH 238 — Differential Equations I
final exam: Monday, May 5, 2025

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. The final exam will be comprehensive, that is, it will cover the material of the whole semester.

(a) Do the practice problems for both midterms.

(b) Retake both midterm exams.

(c) Do the problems below. (Solutions are posted.)

Problem 2.

(a) Determine the Laplace transform $\mathcal{L}(2e^{4t} - 6e^{-t} + 3)$.

(b) Determine the Laplace transform $\mathcal{L}((2t-5)e^{-3t})$.

(c) Given $f(t) = \begin{cases} 2e^{3t}, & \text{if } t \geqslant \pi, \\ 0, & \text{otherwise,} \end{cases}$ determine its Laplace transform $\mathcal{L}(f(t))$.

(d) Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{s-16}{s^2-2s-8}\right)$.

Solution.

(a) $\mathcal{L}(2e^{4t} - 6e^{-t} + 3) = 2\mathcal{L}(e^{4t}) - 6\mathcal{L}(e^{-t}) + 3\mathcal{L}(1) = \frac{2}{s-4} - \frac{6}{s+1} + \frac{3}{s}$

(b) $\mathcal{L}((2t-5)e^{-3t}) = 2\mathcal{L}(te^{-3t}) - 5\mathcal{L}(e^{-3t}) = \frac{2}{(s+3)^2} - \frac{5}{s+3}$

Here, we combined $\mathcal{L}(tf(t)) = -F'(s)$ with $\mathcal{L}(e^{-3t}) = \frac{1}{s+3}$ to get $\mathcal{L}(te^{-3t}) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s+3} = \frac{1}{(s+3)^2}$.

(Alternatively, we can combine $\mathcal{L}(2t-5) = \frac{2}{s^2} - \frac{5}{s}$ and $\mathcal{L}(f(t)e^{-3t}) = F(s+3)$ to again get $\mathcal{L}((2t-5)e^{-3t}) = \frac{2}{(s+3)^2} - \frac{5}{s+3}$.)

(c) We can write $f(t) = 2e^{3t}u_\pi(t) = g(t-\pi)u_\pi(t)$ with $g(t-\pi) = 2e^{3t}$ or, equivalently, $g(t) = 2e^{3(t+\pi)} = 2e^{3\pi}e^{3t}$. The latter has Laplace transform $G(s) = \frac{2e^{3\pi}}{s-3}$. Therefore, $\mathcal{L}(f(t)) = \mathcal{L}(u_\pi(t)g(t)) = e^{-\pi s}G(s) = e^{-\pi s}\frac{2e^{3\pi}}{s-3} = \frac{2e^{(3-s)\pi}}{s-3}$.

(d) Note that $s^2 - 2s - 8 = (s+2)(s-4)$. We use partial fractions to write $\frac{s-16}{(s+2)(s-4)} = \frac{A}{s+2} + \frac{B}{s-4}$. We find the coefficients as

$$A = \left.\frac{s-16}{s-4}\right|_{s=-2} = 3, \quad B = \left.\frac{s-16}{s+2}\right|_{s=4} = -2.$$

Hence $\mathcal{L}^{-1}\left(\frac{s-16}{s^2-2s-8}\right) = \mathcal{L}^{-1}\left(\frac{3}{s+2} - \frac{2}{s-4}\right) = 3e^{-2t} - 2e^{4t}$.

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Problem 3. Determine the Laplace transform of the unique solutions to the following initial value problems.

[Do not determine the solution and do not simplify.]

(a) $y'' + 4y' - 3y = 2e^{-4t} + 5t^2,\ y(0) = 7,\ y'(0) = -2.$

(b) $y'' - 6y' + 5y = \begin{cases} 3, & \text{if } 1 \leqslant t < 4, \\ 0, & \text{otherwise,} \end{cases},\ y(0) = 2,\ y'(0) = -1.$

(c) $y_1' = 3y_1 - 4y_2 - 5t^2 e^{-3t},\ y_2' = y_1 + 2y_2,\ y_1(0) = 3,\ y_2(0) = -2.$

Solution.

(a) The DE $y'' + 4y' - 3y = 2e^{-4t} + 5t^2$ transforms into

$$s^2 Y - sy(0) - y'(0) + 4(sY - y(0)) - 3Y = (s^2 + 4s - 3)Y - 7s - 26 = \frac{2}{s+4} + \frac{10}{s^3}.$$

Accordingly, $Y(s) = \frac{1}{s^2 + 4s - 3}\left[\frac{2}{s+4} + \frac{10}{s^3} + 7s + 26\right]$ is the Laplace transform of the unique solution to the IVP.

(b) First, we observe that the right-hand side of the differential equation can be written as $3(u_1(t) - u_4(t))$. It follows from the Laplace transform table that $\mathcal{L}(u_a(t)) = e^{-as}\frac{1}{s}$ (using the entry for $u_a(t)f(t-a)$ with $f(t) = 1$). Consequently, $\mathcal{L}(3(u_1(t) - u_4(t))) = 3e^{-s}\frac{1}{s} - 3e^{-4s}\frac{1}{s} = \frac{3}{s}(e^{-s} - e^{-4s})$.

Taking the Laplace transform of both sides of the DE, we therefore get

$$s^2 Y(s) - sy(0) - y'(0) - 6(sY(s) - y(0)) + 5Y(s) = \frac{3}{s}(e^{-s} - e^{-4s}),$$

which using the initial values simplifies to

$$(s^2 - 6s + 5)Y(s) - 2s + 1 + 6 \cdot 2 = \frac{3}{s}(e^{-s} - e^{-4s}).$$

We conclude that the Laplace transform of the unique solution is

$$Y(s) = \frac{1}{s^2 - 6s + 5}\left[\frac{3}{s}(e^{-s} - e^{-4s}) + 2s - 13\right].$$

(c) $y_1' = 3y_1 - 4y_2 - 5t^2 e^{-3t}$ transforms into $sY_1 - \underbrace{y_1(0)}_{=3} = 3Y_1 - 4Y_2 - \frac{10}{(s+3)^3}$ or $(s-3)Y_1 + 4Y_2 = 3 - \frac{10}{(s+3)^3}$.

Likewise, $y_2' = y_1 + 2y_2$ transforms into $sY_2 - \underbrace{y_2(0)}_{=-2} = Y_1 + 2Y_2$.

By the second equation, $Y_1 = (s-2)Y_2 + 2$. Used in the first, we get $(s-3)((s-2)Y_2 + 2) + 4Y_2 = 3 - \frac{10}{(s+3)^3}$ or $((s-3)(s-2) + 4)Y_2 = 3 - \frac{10}{(s+3)^3} - 2(s-3)$.

Hence, $Y_2 = \left(3 - \frac{10}{(s+3)^3} - 2(s-3)\right) / ((s-3)(s-2) + 4)$.

Correspondingly, $Y_1 = (s-2)Y_2 + 2 = (s-2)\left(3 - \frac{10}{(s+3)^3} - 2(s-3)\right) / ((s-3)(s-2) + 4) + 2$.

Comment. The important thing is to realize that, after taking the Laplace transform of the two differential equations, we obtain two ordinary (linear) equations in Y_1 and Y_2 that we can easily solve (although the formulas become involved). If we wanted to compute the solutions $y_i(t)$, we could then do a partial fraction decomposition of the rational functions for Y_i (both the involved roots would be irrational and the result not pretty).

For the final exam, you will be provided the following table of Laplace transforms.

$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
e^{at}	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{at}f(t)$	$F(s-a)$
$t f(t)$	$-F'(s)$
$u_a(t)f(t-a)$	$e^{-as}F(s)$

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