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MATH 286 SECTION G1 – Introduction to Differential Equations Plus

MIDTERM EXAMINATION 2

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**INSTRUCTIONS**

- This exam 50 minutes long. No personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- EXPLAIN YOUR WORK! Little or no points will be given for a correct answer with no explanation of how you got it. If you use a theorem to answer a question, indicate which theorem you are using, and explain why the hypotheses of the theorem are valid.
- GOOD LUCK!

**PLEASE NOTE:** "Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written."

| Question: | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Points: | 15 | 15 | 12 | 12 | 54 |
| Score: | | | | | |

1. Consider the ordinary differential equation

$$Ly = y^{(3)} - 2y'' + 2y' = x + xe^x.$$

(a) (5 points) Determine the complementary function $y_c(x)$ for the above ODE.

**Solution:** The characteristic polynomial is

$$P(r) = r^3 - 2r^2 + 2r = r(r^2 - 2r + 2) = r(r-1-i)(r-1+i).$$

Therefore the complementary function (which is the solution to $Ly = 0$) is

$$y_c(x) = c_1 + c_2 e^x \cos x + c_2 e^x \sin x.$$

(b) (8 points) Find a particular solution $y_p(x)$ to the above ODE.

**Solution:** Since $F(x) = x + xe^x$, a first guess at a trial solution would be of the form $Ax+B+(Cx+D)e^x$. But there is overlap with the complementary function $y_c(x)$ coming from the constant $B$. To eliminate this we multiply $Ax + B$ by $x$ and get a particular solution of the form

$$\begin{aligned}
& y_p(x) = x(Ax+B) + (Cx+D)e^x \\
\implies & y_p'(x) = 2Ax + B + Ce^x + Cxe^x + De^x = 2Ax + B + Cxe^x + (C+D)e^x \\
\implies & y_p''(x) = 2A + (C+D)e^x + Ce^x + Cxe^x = 2A + Cxe^x + (2C+D)e^x \\
\implies & y_p^{(3)}(x) = (2C+D)e^x + Ce^x + Cxe^x = Cxe^x + (3C+D)e^x.
\end{aligned}$$

Setting $Ly_p = F$ and comparing coefficients, we get

$$\begin{aligned}
(\text{coefficient of } xe^x) &: 1 = C - 2C + 2C \implies C = 1 \\
(\text{coefficient of } e^x) &: 0 = 3C + D - 2(2C+D) + 2(C+D) \implies D = -1 \\
(\text{coefficient of } x) &: 1 = 4A \implies A = \frac{1}{4} \\
(\text{coefficient of } 1) &: 0 = -4A + 2B \implies B = \frac{1}{2}.
\end{aligned}$$

(c) (2 points) Write down the general solution to the above ODE.

**Solution:** The general solution is

$$y(x) = y_c(x) + y_p(x) = c_1 + c_2 e^x \cos x + c_2 e^x \sin x + \frac{1}{4}x^2 + \frac{1}{2}x + (x-1)e^x$$

2. Consider the ordinary differential equation

$$Ly = y'' - \frac{1}{x}y' + \frac{1}{x^2}y = x^3.$$

(a) (4 points) Verify that $y_1(x) = x$ and $y_2(x) = x\ln|x|$ are solutions to the associated homogeneous ODE

$$Ly = 0 \qquad (x \neq 0).$$

**Solution:**

$$y_1'' - \frac{1}{x}y_1' + \frac{1}{x^2}y_1 = 0 - \frac{1}{x} + \frac{1}{x} = 0.$$

$$y_2'' - \frac{1}{x}y_2' + \frac{1}{x^2}y_2 = \frac{1}{x} - \frac{1}{x}(\ln|x|+1) + \frac{1}{x}\ln|x| = 0.$$

(b) (5 points) Compute the Wronskian $W(y_1, y_2)$ for the pair of functions $y_1, y_2$ above. Are $y_1$ and $y_2$ linearly independent on the interval $I = (0, \infty)$? Why or why not?

**Solution:**

$$W(x) = \det\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = x(\ln|x|+1) - 1(x\ln|x|) = x.$$

Since $y_1, y_2$ are solutions to the ODE $y'' + p(x)y' + q(x) = 0$ and $W(y_1, y_2) \neq 0$ on $I$, the theorem on linear independence and Wronskians implies that these functions are linearly independent on $I$.

(c) (6 points) Find a particular solution to

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = x^3 \qquad (x > 0).$$

(**Hint:** One possible approach is to use the variation of parameters method.)

**Solution:** Let

$$u_1(x) = -\int \frac{y_2(x)x^3}{W(x)}dx = -\int x^3\ln|x|dx = \frac{-x^4}{4}\ln|x| + \int \frac{x^3}{4}dx = \frac{-x^4}{4}\ln|x| + \frac{x^4}{16}.$$

$$u_2(x) = \int \frac{y_1(x)x^3}{W(x)}dx = \int x^3 dx = \frac{x^4}{4}.$$

Then the method of variation of parameters gives

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = \frac{x^5}{4}\left(\frac{1}{4} - \ln|x|\right) + \frac{x^5}{4}\ln|x| = \frac{x^5}{16}.$$

3. (12 points) For the following endpoint problem, determine all eigenvalues $\lambda \in \mathbb{R}$ and their associated eigenfunctions:

$$y'' - 4y' + \lambda y = 0, \qquad y(0) = y(1) = 0.$$

**Solution:** The characteristic equation is

$$0 = P(r) = r^2 - 4r + \lambda \iff r = 2 \pm \frac{\sqrt{16 - 4\lambda}}{2} = 2 \pm \sqrt{4 - \lambda}.$$

**Case 1:** If $\lambda < 4$, the general solution is

$$y(t) = c_1 e^{(2+\sqrt{4-\lambda})t} + c_2 e^{(2-\sqrt{4-\lambda})t}.$$

Plugging in the endpoint conditions, we find $c_1 = c_2 = 0$. So, there are no eigenvalues $\lambda < 4$.

**Case 2:** If $\lambda = 4$, the general solution is

$$y(t) = c_1 e^{2t} + c_2 t e^{2t}.$$

Plugging in the endpoint conditions, we find

$$0 = y(0) = c_1 \qquad 0 = y(1) = c_2 e^2 \implies c_1 = c_2 = 0.$$

So $\lambda = 4$ is not an eigenvalue.

**Case 3:** If $\lambda > 4$, the general solution is

$$y(t) = c_1 e^{2t} \cos \sqrt{\lambda - 4}t + c_2 e^{2t} \sin \sqrt{\lambda - 4}t.$$

The endpoint conditions then give

$$0 = y(0) = c_1 \qquad 0 = y(1) = c_2 e^2 \sin \sqrt{\lambda - 4} \implies \sqrt{\lambda - 4} = n\pi \quad (n = 1, 2, 3, \ldots).$$

So the eigenvalues are

$$\lambda_n = 4 + n^2\pi^2 \qquad (n = 1, 2, \ldots),$$

and the eigenfunctions are

$$y_n(t) = e^{2t} \sin(n\pi t)$$

4. Consider a mass-spring-dashpot system with mass $m = 1$kg, spring constant $k = 4$ N/m and dashpot damping constant $\beta > 0$ N s/m. Let $x(t)$ denote the displacement (in metres, at time $t$) of the mass from its equilibrium resting position.

   (a) (4 points) For what values of $\beta$ is the system underdamped?

   **Solution:** The characteristic polynomial for this system is

   $$P(r) = mr^2 + \beta r + k = r^2 + \beta r + 4,$$

   which has roots $r = \frac{-\beta}{2} \pm \frac{\sqrt{\beta^2-16}}{2}$. The system is underdamped when the roots are complex. I.e., when $\beta^2 - 16 < 0 \iff 0 < \beta < 4$.

   For the remainder of the problem, assume that the dashpot is **disconnected** from the system (i.e., set $\beta = 0$) and that an external force $F(t) = 2\sin\omega t$ Newtons is applied to the mass.

   (b) (2 points) At what forcing frequency $\omega$ will resonance occur in the forced system?

   **Solution:** Resonance will occur when $\omega = \omega_0 = \sqrt{k/m} = 2$, which is the natural frequency of the system.

   (c) (6 points) Write down the general solution $x(t)$ in this case.

   **Solution:** The complementary function for this equation is

   $$x_c(t) = C\cos(2t - \alpha).$$

   A particular solution will be of the form $x_p(t) = At\cos(2t) + Bt\sin(2t)$, setting $y_p'' + 4y_p = 2\sin(2t)$, we get

   $$2\sin(2t) = -4A\sin(2t) - 4At\cos(2t) + 4B\cos(2t) - 4Bt\sin(2t) + 4At\cos(2t) + 4Bt\sin(2t),$$

   giving $A = -1/2$ and $B = 0$. Thus the general solution is

   $$x(t) = x_c(t) + x_p(t) = C\cos(2t - \alpha) - \frac{t}{2}\cos(2t).$$

(*Extra work space.*)