Remark 132. (April Fools' Day!) $1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i\,i = -1$.

When using the principal square-root (which basically takes the positive root, that is, the one with positive real part), the rule $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$ does not hold universally (so the trouble lies with the third equality). It does hold if, for instance, $a \geqslant 0$ or $b \geqslant 0$. Apparently, this trouble historically resulted in lots of controversy around complex numbers, with some mathematicians rejecting them outright; though unjust, it has even been claimed[24] that Euler was confused about the law $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$. $\diamondsuit$

Review. Consider $\boldsymbol{x}' = A\boldsymbol{x}$, as usual.

- Any fundamental matrix $\Phi(t)$ satisfies $\Phi' = A\Phi$.
- e^{At} is a fundamental matrix. It is the unique fundamental matrix $\Phi(t)$ with $\Phi(0) = I$. $\diamondsuit$

The matrix exponential has some nice properties:

- $e^{A+B} = e^A e^B$ if $AB = BA$.

  This can be shown starting with the series $e^{A+B} = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!}$ by expanding $(A+B)^n$ using the binomial theorem. For instance, $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$. In order to simplify this to $(A+B)^2 = A^2 + 2AB + B^2$, we need $AB = BA$.
- $(e^A)^{-1} = e^{-A}$

  This follows from the previous upon setting $B = -A$.

Theorem 133. If $\Phi(t)$ is any fundamental matrix, then $e^{At} = \Phi(t)\Phi(0)^{-1}$. [see also Lecture 33]

Proof. Both sides are fundamental matrices (why is $\Phi(t)\Phi(0)^{-1}$ another fundamental matrix?!) and, hence, satisfy the IVP $\Phi' = A\Phi$, $\Phi(0) = I$. By uniqueness, they have to be equal. $\square$

Example 134. Compute e^{At} for $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$.

Solution. The characteristic polynomial is $(2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$.
Hence, the eigenvalues are $\lambda = -1, 4$.

$\lambda = -1$: $\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}\boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $\boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$\lambda = 4$: $\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}\boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $\boldsymbol{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

This gives us the fundamental matrix $\Phi(t) = \begin{pmatrix} e^{-t} & 3e^{4t} \\ -e^{-t} & 2e^{4t} \end{pmatrix}$.

$\Phi(0) = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$, $\Phi(0)^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$

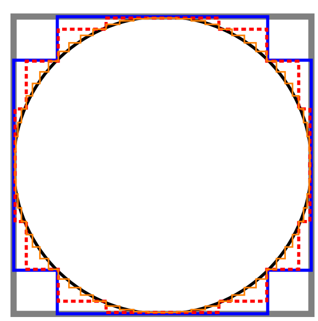
Hence, $e^{At} = \Phi(t)\Phi(0)^{-1} = \frac{1}{5}\begin{pmatrix} 2e^{-t} + 3e^{4t} & -3e^{-t} + 3e^{4t} \\ -2e^{-t} + 2e^{4t} & 3e^{-t} + 2e^{4t} \end{pmatrix}$.

Note that $e^{At}|_{t=0} = I$, indeed. $\diamondsuit$

Theorem 135. The IVP $\boldsymbol{x}' = A\boldsymbol{x}$, $\boldsymbol{x}(0) = \boldsymbol{x}_0$ has solution $\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0$.

Proof. The general solution is $\boldsymbol{x}(t) = e^{At}\boldsymbol{c}$. $\boldsymbol{x}(0) = e^{At}\boldsymbol{c}|_{t=0} = \boldsymbol{c}$. To solve the IVP, we choose $\boldsymbol{c} = \boldsymbol{x}_0$. $\square$

Remark 136. (April Fools' Day!) π is the perimeter of a circle enclosed in a square with edge length 1. The perimeter of the square is 4, which approximates π. To get a better approximation, we "fold" the vertices of the square towards the circle (and get the blue polygon). This construction can be repeated for even better approximations and, in the limit, our shape will converge to the true circle. At each step, the perimeter is 4, so we conclude that $\pi = 4$, contrary to popular belief. $\diamondsuit$



24. See, for instance: `https://webspace.utexas.edu/aam829/1/m/Euler_files/EulerMonthly.pdf`

Armin Straub
astraub@illinois.edu