Review. Useful substitutions ◇

Example 27. Solve $(x-y)y' = x+y$.

Solution. Divide the DE by x to get $\left(1-\frac{y}{x}\right)y' = 1+\frac{y}{x}$. This is a homogeneous equation!
We therefore substitute $u = \frac{y}{x}$, to find the new DE $xu' + u = \frac{1+u}{1-u}$, $xu' = \frac{1+u^2}{1-u}$.
Separation of variable: $\frac{1-u}{1+u^2}\mathrm{d}u = \frac{1}{x}\mathrm{d}x$, $\arctan(u) - \frac{1}{2}\ln(1+u^2) = \ln|x| + C$.
Setting $u = y/x$, we get the (general) implicit solution $\arctan(y/x) - \frac{1}{2}\ln(1+(y/x)^2) = \ln|x| + C$. ◇

Example 28. Solve $y'' = x - y'$.

Solution. Substitute $u = y'$. Then $u' = x - u$, which is linear with $u = x - 1 + Ce^{-x}$.
Hence $y = \frac{1}{2}x^2 - x - Ce^{-x} + D$. ◇

Example 29. Find a general solution to $y'' = 2yy'$.

Solution. We substitute $u = y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. Then $y'' = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y}\cdot u$.
Therefore, our DE turns into $u\frac{\mathrm{d}u}{\mathrm{d}y} = 2yu$.
Dividing by u, we get $\frac{\mathrm{d}u}{\mathrm{d}y} = 2y$. [Note that we loose the solution $u = 0$, which gives the singular solution $y = C$.]
Hence, $u = y^2 + C$. It remains to solve $y' = y^2 + C$. This is a separable DE.
$\frac{1}{C+y^2}\mathrm{d}y = \mathrm{d}x$. Let us restrict to $C = D^2 \geqslant 0$ here. (This means we will only find “half” of the solutions.)
$\int \frac{1}{D^2+y^2}\mathrm{d}y = \frac{1}{D^2}\int \frac{1}{1+(y/D)^2}\mathrm{d}y = \frac{1}{D}\arctan(y/D) = x + A$.
Solving for y, we find $y = D\tan(Dx + AD) = D\tan(Dx + B)$. $[B = AD]$ ◇

Some more applications

Acceleration–velocity models

We consider a falling object, and let $y(t)$ be its height at time t.
If we only take earth's gravitation into account, then the fall is modelled by $y''(t) = -g$.
For many applications, one needs to take air resistance into account.

This is actually less well understood than one might think. Reasonable physical assumptions imply that the resistance is proportional to the square of the velocity[4]. However, for “relatively slowly” falling objects one might empirically observe that the resistance is proportional to the velocity itself[5]. Or anything in between...

Example 30. If air resistance is proportional to velocity, then $y''(t) = -\rho y'(t) - g$. Solve this equation with initial conditions $y(0) = y_0$, $y'(0) = v_0$. [Note that $-\rho y' > 0$ because $y' < 0$.]

Solution. Set $v = y'$, get $v' + \rho v = -g$ (linear!). Integrating factor $e^{\rho t}$. $e^{\rho t}v = \int -ge^{\rho t}\mathrm{d}t = -g/\rho e^{\rho t} + C$.
$v(0) = y'(0) = v_0$ implies $C = v_0 + g/\rho$. Hence, $v(t) = (v_0 + g/\rho)e^{-\rho t} - g/\rho$.
$y(t) = \int v(t)\mathrm{d}t = \ldots = -(v_0/\rho + g/\rho^2)(e^{-\rho t} - 1) - gt/\rho + y_0$, where we used that $y(0) = y_0$.

Note that $\lim_{t\to\infty} v(t) = -g/\rho$. Terminal speed g/ρ. (Important for idea behind a parachute!) ◇

4. A simplistic way to think about this is to imagine the falling object to bump into (air) particles; if the object falls twice as fast, then the momentum of the particles it bumps into is twice as large and it bumps into twice as many of them.

5. Maybe it helps to imagine that, at slow speed, the falling object doesn't exactly bump into particles but instead just gently pushes them aside; so that at twice the speed it only needs to gently push twice as often.