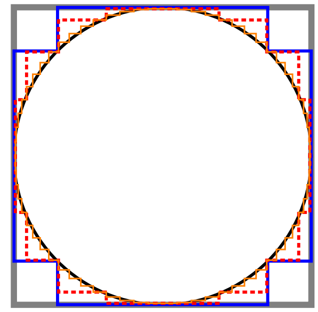
(Halloween scare, revisited) π is the perimeter of a circle enclosed in a square with edge length 1. The perimeter of the square is 4, which approximates π. To get a better approximation, we "fold" the vertices of the square towards the circle (and get the blue polygon). This construction can be repeated for even better approximations and, in the limit, our shape will converge to the true circle. At each step, the perimeter is 4, so we conclude that $\pi = 4$, contrary to popular belief.



Can you pin-point the fallacy in this argument?

(We are not doing something completely silly! For instance, the areas of our approximations do converge to $\pi / 4$, the area of the circle.)

See below for a "solution" to the Halloween scare.

($\pi = 4$, "solution")

We are constructing curves c_n with the property that $c_n \to c$ where c is the circle. This convergence can be understood, for instance, in the same sense $\|c_n - c\| \to 0$ with the norm measuring the maximum distance between the two curves.

Since $c_n \to c$ we then wanted to conclude that $\text{perimeter}(c_n) \to \text{perimeter}(c)$, leading to $4 \to \pi$.

However, in order to conclude from $x_n \to x$ that $f(x_n) \to f(x)$ we need that f is continuous (at x)!!

The "function" perimeter, however, is not continuous. In words, this means that (as we see in this example) curves can be arbitrarily close, yet have very different arc length.

We can dig a little deeper: as we learned in Calculus II, the arc length of a function $y = f_n(x)$ for $x \in [a, b]$ is

$$\int_a^b \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \int_a^b \sqrt{1 + f_n'(x)^2}\mathrm{d}x.$$

Observe that this involves $f_n'(x)$. Try to see why the operator D that sends f to f' is not continuous with respect to the distance induced by the norm

$$\|f\| = \left(\int_a^b f(x)^2 \mathrm{d}x\right)^{1/2}.$$

In words, two functions f and g can be arbitrarily close, yet have very different derivatives f' and g'.

That's a huge issue in **functional analysis**, which is the generalization of linear algebra to infinite dimensional spaces (like the space of all differentiable functions). The linear operators ("matrices") on these spaces frequently fail to be continuous.

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Solving systems of DEs using Laplace transforms

We solved the following system in Example 131 using elimination and our method for solving linear DEs with constant coefficients based on characteristic roots.

Example 171. Solve the system $y_1' = 5y_1 + 4y_2$, $y_2' = 8y_1 + y_2$, $y_1(0) = 0$, $y_2(0) = 1$.

Solution. (using Laplace transforms) $y_1' = 5y_1 + 4y_2$ transforms into $sY_1 - \underbrace{y_1(0)}_{=0} = 5Y_1 + 4Y_2$.

Likewise, $y_2' = 8y_1 + y_2$ transforms into $sY_2 - \underbrace{y_2(0)}_{=1} = 8Y_1 + Y_2$.

The transformed equations are regular equations that we can solve for Y_1 and Y_2.

For instance, by the first equation, $Y_2 = \frac{1}{4}(s-5)Y_1$.

Used in the second equation, we get $\underbrace{-8Y_1 + \frac{1}{4}(s-1)(s-5)Y_1}_{=\frac{1}{4}(s^2-6s-27)=\frac{1}{4}(s+3)(s-9)} = 1$ so that $Y_1 = \frac{4}{(s+3)(s-9)}$.

Hence, the system is solved by $Y_1 = \frac{4}{(s+3)(s-9)}$ and $Y_2 = \frac{1}{4}(s-5)Y_1 = \frac{s-5}{(s+3)(s-9)}$.

As a final step, we need to take the inverse Laplace transform to get $y_1(t) = \mathcal{L}^{-1}(Y_1(s))$ and $y_2(t) = \mathcal{L}^{-1}(Y_2(s))$.

Using partial fractions, $Y_1(s) = \frac{4}{(s+3)(s-9)} = -\frac{1}{3} \cdot \frac{1}{s+3} + \frac{1}{3} \cdot \frac{1}{s-9}$ so that $y_1(t) = -\frac{1}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Similarly, $Y_2(s) = \frac{s-5}{(s+3)(s-9)} = \frac{2}{3} \cdot \frac{1}{s+3} + \frac{1}{3} \cdot \frac{1}{s-9}$ so that $y_2(t) = \frac{2}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Solution. (old solution, for comparison) Since $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1$ (from the first eq.), we have $y_2' = \frac{1}{4}y_1'' - \frac{5}{4}y_1'$.

Using these in the second equation, we get $\frac{1}{4}y_1'' - \frac{5}{4}y_1' = 8y_1 + \frac{1}{4}y_1' - \frac{5}{4}y_1$.

Simplified, this is $y_1'' - 6y_1' - 27y_1 = 0$.

This is a homogeneous linear DE with constant coefficients. The characteristic roots are $-3, 9$.

We therefore obtain $y_1 = C_1e^{-3t} + C_2e^{9t}$ as the general solution.

Thus, $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1 = \frac{1}{4}(-3C_1e^{-3t} + 9C_2e^{9t}) - \frac{5}{4}(C_1e^{-3t} + C_2e^{9t}) = -2C_1e^{-3t} + C_2e^{9t}$.

We determine the (unique) values of C_1 and C_2 using the initial conditions:

$y_1(0) = C_1 + C_2 \overset{!}{=} 0$

$y_2(0) = -2C_1 + C_2 \overset{!}{=} 1$

We solve these two equations and find $C_1 = -\frac{1}{3}$ and $C_2 = \frac{1}{3}$.

The unique solution to the IVP therefore is $y_1(t) = -\frac{1}{3}e^{-3t} + \frac{1}{3}e^{9t}$ and $y_2(t) = \frac{2}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Excursion: Euler's identity

Let's revisit Euler's identity from Theorem 88.

Theorem 172. (Euler's identity) $e^{ix} = \cos(x) + i\sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] ☐

On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Example 173. Where do trig identities like $\sin(2x) = 2\cos(x)\sin(x)$ or $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ (and infinitely many others!) come from?

Short answer: these come from the simple exponential law $e^{x+y} = e^x e^y$.

Let us illustrate this in the simple case $(e^x)^2 = e^{2x}$. Observe that

$$\begin{aligned} e^{2ix} &= \cos(2x) + i\sin(2x) \\ e^{ix}e^{ix} &= [\cos(x) + i\sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i\cos(x)\sin(x). \end{aligned}$$

Comparing imaginary parts (the "stuff with an i"), we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.
Likewise, comparing real parts, we read off $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(Use $\cos^2(x) + \sin^2(x) = 1$ to derive $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ from the last equation.)

Challenge. Can you find a triple-angle trig identity for $\cos(3x)$ and $\sin(3x)$ using $(e^x)^3 = e^{3x}$?
Or, use $e^{i(x+y)} = e^{ix}e^{iy}$ to derive $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x+y) = \ldots$

Realize that the complex number $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ corresponds to the point $(\cos(\theta), \sin(\theta))$.
These are precisely the points on the unit circle!

Recall that a point (x, y) can be represented using **polar coordinates** (r, θ), where r is the distance to the origin and θ is the angle with the x-axis.

Then, $x = r\cos\theta$ and $y = r\sin\theta$.

Every complex number z can be written in **polar form** as $z = re^{i\theta}$, with $r = |z|$.

Why? By comparing with the usual polar coordinates ($x = r\cos\theta$ and $y = r\sin\theta$), we can write

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}.$$

In the final step, we used Euler's identity.

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Application to military strategy: Lanchester's equations

In military strategy, Lanchester's equations can be used to model two opposing forces during "aimed fire" battle.

Let $x(t)$ and $y(t)$ describe the number of troops on each side. Then Lanchester (during World War I) assumed that the rates $-x'(t)$ and $-y'(t)$, at which soldiers are put out of action, are proportional to the number of opposing forces. That is:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\beta y(t) \\ -\alpha x(t) \end{bmatrix}, \quad \text{or, in matrix form: } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -\beta \\ -\alpha & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The proportionality constants $\alpha, \beta > 0$ indicate the strength of the forces ("fighting effectiveness coefficients").

These are simple linear DEs with constant coefficients, which we have learned how to solve.

Comment. The "aimed fire" means that all combatants are engaged, as is common in modern combat with long-range weapons. This is rather different than ancient combat where only some of the soldiers (such as those in front) were engaged at a time. For more details, see: `https://en.wikipedia.org/wiki/Lanchester%27s_laws`

Example 174. Solve Lanchester's equations with initial conditions $x(0) = x_0$ and $y(0) = y_0$.

Solution. (using Laplace transforms) $x' = -\beta y$ transforms into $sX - x_0 = -\beta Y$. Likewise, $y' = -\alpha x$ transforms into $sY - y_0 = -\alpha X$. The transformed equations are regular equations that we can solve for X and Y. For instance, by the first equation, $Y = -\frac{1}{\beta}(sX - x_0)$.

Used in the second equation, we get $-\frac{s}{\beta}(sX - x_0) - y_0 = -\alpha X$ so that $(s^2 - \alpha\beta)X = sx_0 - \beta y_0$.

Hence, the system is solved by $X = \frac{sx_0 - \beta y_0}{s^2 - \alpha\beta}$ and $Y = -\frac{1}{\beta}(sX - x_0) = \frac{sy_0 - \alpha x_0}{s^2 - \alpha\beta}$.

As a final step, we need to take the inverse Laplace transform to get $x(t) = \mathcal{L}^{-1}(X(s))$ and $y(t) = \mathcal{L}^{-1}(Y(s))$.

Using partial fractions, $X(s) = \frac{sx_0 - \beta y_0}{(s - \sqrt{\alpha\beta})(s + \sqrt{\alpha\beta})} = \frac{A}{s - \sqrt{\alpha\beta}} + \frac{B}{s + \sqrt{\alpha\beta}}$ with

$$A = \frac{sx_0 - \beta y_0}{s + \sqrt{\alpha\beta}}\bigg|_{s = \sqrt{\alpha\beta}} = \frac{\sqrt{\alpha\beta}\, x_0 - \beta y_0}{2\sqrt{\alpha\beta}} = \frac{1}{2}\left(x_0 - y_0\sqrt{\frac{\beta}{\alpha}}\right), \quad B = \frac{sx_0 - \beta y_0}{s - \sqrt{\alpha\beta}}\bigg|_{s = -\sqrt{\alpha\beta}} = \frac{1}{2}\left(x_0 + y_0\sqrt{\frac{\beta}{\alpha}}\right).$$

It follows that $x(t) = Ae^{\sqrt{\alpha\beta}\, t} + Be^{-\sqrt{\alpha\beta}\, t}$. We obtain a similar formula for $y(t)$ (with x_0 and y_0 as well as α and β swapped for each other).

Solution. (without Laplace transforms) Our goal is to write down a single DE that only involves, say, $x(t)$. From the first DE, we get $y(t) = -\frac{1}{\beta}x'(t)$. Hence, $y'(t) = -\frac{1}{\beta}x''(t)$. Using that in the second DE, we obtain $-\frac{1}{\beta}x''(t) = -\alpha x(t)$ or, equivalently, $x''(t) - \alpha\beta x(t) = 0$.

Observe that, since $y(t) = -\frac{1}{\beta}x'(t)$, the initial condition $y(0) = y_0$ translates into $x'(0) = -\beta y_0$.

The roots are $\pm r$ where $r = \sqrt{\alpha\beta}$. Hence, $x(t) = C_1 e^{rt} + C_2 e^{-rt}$.

Using the initial conditions $x(0) = x_0$ and $x'(0) = -\beta y_0$, we find $C_1 + C_2 = x_0$ and $rC_1 - rC_2 = -\beta y_0$.

This results in $C_1 = \frac{1}{2}\left(x_0 - \frac{\beta y_0}{r}\right)$ and $C_2 = \frac{1}{2}\left(x_0 + \frac{\beta y_0}{r}\right)$. Correspondingly, using $r = \sqrt{\alpha\beta}$,

$$x(t) = \frac{1}{2}\left(x_0 - y_0\sqrt{\frac{\beta}{\alpha}}\right)e^{\sqrt{\alpha\beta}\, t} + \frac{1}{2}\left(x_0 + y_0\sqrt{\frac{\beta}{\alpha}}\right)e^{-\sqrt{\alpha\beta}\, t}$$

with a similar formula for $y(t) = -\frac{1}{\beta}x'(t)$.

Comment. The formulas take a particularly pleasing form when written in terms of $\cosh$ and $\sinh$ instead:

$$x(t) = x_0 \cosh(\sqrt{\alpha\beta}\, t) - y_0\sqrt{\frac{\beta}{\alpha}} \sinh(\sqrt{\alpha\beta}\, t), \qquad y(t) = y_0 \cosh(\sqrt{\alpha\beta}\, t) - x_0\sqrt{\frac{\alpha}{\beta}} \sinh(\sqrt{\alpha\beta}\, t).$$

Example 175. Determine conditions on x_0, y_0 (size of forces) and α, β (effectiveness of forces) that allow us to conclude who will win the battle.

Solution. Instead of analyzing our explicit formulas to find out which of $x(t)$ and $y(t)$ becomes 0 first (and therefore loses the battle), we make the following mathematical observation: the DEs dictate that, while fighting, both $x(t)$ and $y(t)$ are decreasing. On the other hand, purely mathematically, once one of the two turns negative then the DEs dictate that the other will increase while the negative one continues decreasing. Therefore, a force wins when its mathematical formula is increasing for large t.

Both solutions are combinations of $e^{\sqrt{\alpha\beta}t}$ and $e^{-\sqrt{\alpha\beta}t}$. Clearly, the term $e^{\sqrt{\alpha\beta}t}$ dominates the other as t gets large. For $x(t)$ that term has coefficient $\frac{1}{2}(x_0 - y_0\sqrt{\beta/\alpha})$. This allows us to conclude that $x(t)$ wins the battle if $x_0 - y_0\sqrt{\frac{\beta}{\alpha}} > 0$. This is equivalent to $\alpha x_0^2 > \beta y_0^2$.

Solution. (without solving the DE) As an alternative, we can also start fresh and divide the two equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\beta y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -\alpha x$$

to get $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\alpha x}{\beta y}$. Using separation of variables, we find $\beta y \,\mathrm{d}y = \alpha x \,\mathrm{d}x$ which implies $\frac{1}{2}\beta y^2 = \frac{1}{2}\alpha x^2 + D$.
Consequently, $\alpha x^2 - \beta y^2 = C$ where $C = -2D$ is a constant. Using the initial conditions, we find $C = \alpha x_0^2 - \beta y_0^2$.
If $y(t_1) = 0$ (meaning that x wins at time t_1), then $\alpha x(t_1)^2 = C > 0$. On the other hand, if $x(t_1) = 0$, then $-\beta y(t_1)^2 = C < 0$. In other words, the sign of C determines who will win the battle.
Namely, x will win if $C > 0$ which is equivalent to $\alpha x_0^2 > \beta y_0^2$.

Conclusion. The condition we found is known as **Lanchester's square law**: its crucial message is that the sizes x_0, y_0 of the forces count quadratically, whereas the fighting effectivenesses α, β only count linearly. In other words, to beat a force with twice the effectiveness the other side only needs to have a force that is about 41.4% larger (since $\sqrt{2} \approx 1.4142$). Or, put differently, to beat a force of twice the size, the other side would need a fighting effectiveness that is more than 4 times as large.

Application to epidemiology: SIR model

The next example application results in a system of **nonlinear** differential equations. We do not have the tools to solve such equations.

Example 176. (epidemiology) Let us indicate the popular SIR model for short outbreaks of diseases among a population of constant size N.

In a SIR model, the population is compartmentalized into $S(t)$ susceptible, $I(t)$ infected and $R(t)$ recovered (or resistant) individuals ($N = S(t) + I(t) + R(t)$). In the Kermack-McKendrick model, the outbreak of a disease is modeled by

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I,$$

with γ modeling the recovery rate and β the infection rate. Note that this is a non-linear system of differential equations. For more details and many variations used in epidemiology, see:
https://en.wikipedia.org/wiki/Compartmental_models_in_epidemiology

Comment. The following variation

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I R, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I R,$$

which assumes "infectious recovery", was used in 2014 to predict that facebook might lose 80% of its users by 2017. It is that claim, not mathematics (or even the modeling), which attracted a lot of media attention.
http://blogs.wsj.com/digits/2014/01/22/controversial-paper-predicts-facebook-decline/