Example 163. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right)$.

Solution. $\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right) = e^{3t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = te^{3t}$.

Example 164. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{(s-3)^2}\right)$.

Solution. The previous example and $\mathcal{L}(e^{at}f(t)) = F(s-a)$ imply that $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{(s-3)^2}\right) = u_2(t)(t-2)\,e^{3(t-2)}$.

Example 165. Solve the IVP $y'' - 3y' + 2y = e^t$, $y(0) = 0$, $y'(0) = 1$.

Solution. (old style, outline) The characteristic polynomial $D^2 - 3D + 2 = (D-1)(D-2)$. Since there is duplication, we have to look for a particular solution of the form $y_p = Ate^t$. To determine A, we need to plug into the DE (we find $A = -1$). Then, the general solution is $y(t) = Ate^t + C_1e^t + C_2e^{2t}$, and the initial conditions determine C_1 and C_2 (we find $C_1 = -2$ and $C_2 = 2$).

Solution. (Laplace style)

$$\begin{aligned}
\mathcal{L}(y''(t)) - 3\mathcal{L}(y'(t)) + 2\mathcal{L}(y(t)) &= \mathcal{L}(e^t) \\
s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) &= \frac{1}{s-1} \\
(s^2 - 3s + 2)Y(s) &= 1 + \frac{1}{s-1} = \frac{s}{s-1} \\
Y(s) &= \frac{s}{(s-1)^2(s-2)}
\end{aligned}$$

At this point, we have found the Laplace transform $Y(s)$ of the solution $y(t)$.

To find $y(t)$, we again use partial fractions. We find $Y(s) = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}$ with coefficients

$$C = \left.\frac{s}{(s-1)^2}\right|_{s=2} = 2, \quad A = \left.\frac{s}{s-2}\right|_{s=1} = -1, \quad B = \left.\frac{\mathrm{d}}{\mathrm{d}s}\frac{s}{s-2}\right|_{s=1} = \left.\frac{-2}{(s-2)^2}\right|_{s=1} = -2.$$

Finally, $y(t) = \mathcal{L}^{-1}\left(\frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}\right) = Ate^t + Be^t + Ce^{2t} = -(t+2)e^t + 2e^{2t}$.

More details on the partial fractions with a repeated root. Above we computed A, B, C so that

$$\frac{s}{(s-1)^2(s-2)} = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}.$$

- We can compute C as before by multiplying both sides with $s-2$ and then setting $s = 2$.
- Similarly, we can compute A by multiplying both sides with $(s-1)^2$ and then setting $s = 1$.
- To compute B, multiply both sides by $(s-1)^2$ (as for A) to get $\frac{s}{(s-2)} = A + B(s-1) + \frac{C(s-1)^2}{s-2}$.

  Now, we take the derivative on both sides (so that A goes away) to get

  $$\frac{-2}{(s-2)^2} = B + \frac{C(2(s-1)(s-2) - (s-1)^2)}{(s-2)^2}$$

  and we find B by setting $s = 1$.

  Comment. In fact, the term involving C had to drop out when plugging in $s = 1$, even after taking a derivative. That's because, after multiplying with $(s-1)^2$, that term has a double root at $s = 1$. When taking a derivative, it therefore still has a (single) root at $s = 1$.

Armin Straub
straub@southalabama.edu

Hyperbolic sine and cosine

Review. Euler's formula states that $e^{it} = \cos(t) + i\sin(t)$.

Recall that a function $f(t)$ is **even** if $f(-t) = f(t)$. Likewise, it is **odd** if $f(-t) = -t$.

Note that $f(t) = t^n$ is even if and only if n is even. Likewise, $f(t) = t^n$ is odd if and only if n is odd. That's where the names are coming from.

Any function $f(t)$ can be decomposed into an even and an odd part as follows:

$$f(t) = f_{\text{even}}(t) + f_{\text{odd}}(t), \quad f_{\text{even}}(t) = \frac{1}{2}(f(t) + f(-t)), \quad f_{\text{odd}}(t) = \frac{1}{2}(f(t) - f(-t)).$$

Verify that $f_{\text{even}}(t)$ indeed is even, and that $f_{\text{odd}}(t)$ indeed is an odd function (regardless of $f(t)$).

Example 166. The **hyperbolic cosine**, denoted $\cosh(t)$, is the even part of e^t. Likewise, the **hyperbolic sine**, denoted $\sinh(t)$, is the odd part of e^t.

- Equivalently, $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$ and $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$.
- In particular, $e^t = \cosh(t) + \sinh(t)$.

  As recalled above, any function is the sum of its even and odd part.

  Comparing with Euler's formula, we find $\cosh(it) = \cos(t)$ and $\sinh(it) = i\sin(t)$. This indicates that $\cosh$ and $\sinh$ are related to $\cos$ and $\sin$, as their name suggests (see below for the "hyperbolic" part).
- $\frac{\mathrm{d}}{\mathrm{d}t}\cosh(t) = \sinh(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\sinh(t) = \cosh(t)$.
- $\cosh(t)$ and $\sinh(t)$ both satisfy the DE $y'' = y$.

  We can write the general solution as $C_1 e^t + C_2 e^{-t}$ or, if useful, as $C_1\cosh(t) + C_2\sinh(t)$.
- $\cosh(t)^2 - \sinh(t)^2 = 1$

  Verify this by substituting $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$ and $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$.

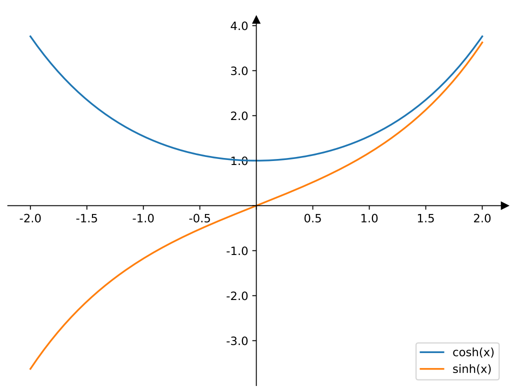
  Note that the equation $x^2 - y^2 = 1$ describes a **hyperbola** (just like $x^2 + y^2 = 1$ describes a circle).

  The above equation is saying that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cosh(t) \\ \sinh(t) \end{bmatrix}$ is a parametrization of the hyperbola.

  Comment. Circles and hyperbolas are conic sections (as are ellipses and parabolas).

  Comment. Hyperbolic geometry plays an important role, for instance, in special relativity:
  https://en.wikipedia.org/wiki/Hyperbolic_geometry

Homework. Write down the parallel properties of cosine and sine.



Armin Straub
straub@southalabama.edu

Example 167. Determine the Laplace transform of $\cosh(\omega t)$ and $\sinh(\omega t)$.

Solution. $\mathcal{L}(\cosh(\omega t)) = \mathcal{L}(\frac{1}{2}(e^{\omega t} + e^{-\omega t})) = \frac{1}{2}\left(\frac{1}{s-\omega} + \frac{1}{s+\omega}\right) = \frac{s}{s^2-\omega^2}$

$\mathcal{L}(\sinh(\omega t)) = \mathcal{L}(\frac{1}{2}(e^{\omega t} - e^{-\omega t})) = \frac{1}{2}\left(\frac{1}{s-\omega} - \frac{1}{s+\omega}\right) = \frac{\omega}{s^2-\omega^2}$

For comparison. We have already seen that $\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$ and $\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$. This once more illustrates that $\cosh$ and $\sinh$ are related to $\cos$ and $\sin$ by $\cosh(it) = \cos(t)$ and $\sinh(it) = i\sin(t)$.

Example 168. Write down a homogeneous linear differential equation satisfied by $y(x) = 5x^2 - 3\cosh(2x)$.

Comment. This is the same as finding an operator $p(D)$ such that $p(D)y = 0$.

Solution. In order for $y(x)$ to be a solution of $p(D)y = 0$, the characteristic roots must include $0, 0, 0, \pm 2$ (note that $\cosh(2x) = \frac{1}{2}(e^{2x} + e^{-2x})$ which contributes the roots ± 2).

Hence, the simplest differential equation is $D^3(D-2)(D+2)y = 0$.

Comment. This is an order 5 differential equation. If we wanted to, we could multiply out $D^3(D-2)(D+2) = D^3(D^2-4) = D^5 - 4D^3$ and write the differential equation in the "classical" form $y^{(5)} - 4y''' = 0$. However, there is typically no benefit in doing so because it is usually more useful to have the DE in factored form (so that the characteristic roots can just be read off). In general, only multiply out factored expressions if there is something to be gained from doing so!

Example 169. A homogeneous linear differential equation with constant coefficients is solved by $y(x) = 2e^{-7x}\cos(3x) - 5x\sinh(4x)$. Which characteristic roots must the DE have?

Solution. The characteristic roots of the differential equation must include $-7 \pm 3i, \pm 4, \pm 4$.

Example 170. Consider the DE $y'' - 2y' + y = 2x\sinh(3x) + 7x^2$. What is the simplest form (with undetermined coefficients) of a particular solution?

Solution. Since $D^2 - 2D + 1 = (D-1)^2$, the characteristic roots are $1, 1$. The roots for the inhomogeneous part are $\pm 3, \pm 3, 0, 0, 0$. Hence, there has to be a particular solution of the form $y_p = (A_1 + A_2 x)\cosh(3x) + (A_3 + A_4 x)\sinh(3x) + A_5 + A_6 x + A_7 x^2$.

(We can then plug into the DE to determine the (unique) values of the coefficients $A_1, A_2, \ldots, A_7$.)

Comment. If we prefer, we can, of course, also express $\sinh(3x)$ in terms of exponentials. Then the DE becomes $y'' - 2y' + y = xe^{3x} - xe^{-3x} + 7x^2$. The characteristic roots of the DE remain the same. The simplest form of a particular solution now is $y_p = (B_1 + B_2 x)e^{3x} + (B_3 + B_4 x)e^{-3x} + B_5 + B_6 x + B_7 x^2$. Make sure that you see that this is equivalent to our earlier form using $\cosh$ and $\sinh$.

Armin Straub
straub@southalabama.edu

Bonus: The fin equation from thermodynamics

The following is an example from thermodynamics. The governing differential equation is a second-order DE that is like the equation describing the motion of a mass on a spring ($my'' + ky = 0$) except that one term has the opposite sign. Besides showcasing an application, we want to show off how $\cosh$ and $\sinh$ are useful for writing certain solutions in a more pleasing form.

Let $T(x)$ describe the temperature at position x in a fin with fin base at $x = 0$ and fin tip at $x = L$.

For more context on fins: `https://en.wikipedia.org/wiki/Fin_(extended_surface)`

If we write $\theta(x) = T(x) - T_\infty$ for the temperature excess at position x (with T_∞ the external temperature), then we find (under various simplifying assumptions) that the temperature distribution in our fin satisfies the following DE, known as the **fin equation**:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - m^2\theta = 0, \quad m^2 = \frac{hP}{kA} > 0.$$

- A is the cross-sectional area of the fin (assumed to be the same for all positions x).
- P is the perimeter of the fin (assumed to be the same for all positions x).
- k is the thermal conductivity of the material (assumed to be constant).
- h is the convection heat transfer coefficient (assumed to be constant).

Since the DE is homogeneous and linear with characteristic roots $\pm m$, the general solution is

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} = D_1 \cosh(mx) + D_2 \sinh(mx).$$

The constants C_1, C_2 (or, equivalently, D_1, D_2) can then be found by imposing appropriate boundary conditions at the **fin base** ($x = 0$) and at the **fin tip** ($x = L$).

In practice, we often know the temperature at the fin base and therefore the temperature excess, resulting in the boundary condition $\theta(0) = \theta_0$. At the fin tip, common boundary conditions are:

- $\theta(L) \to 0$ as $L \to \infty$ (infinitely long fin)

  In this case, the fin is so long that the temperature at the fin tip approaches the external temperature. Mathematically, we get $\theta(x) = Ce^{-mx}$ since $e^{mx} \to \infty$ as $x \to \infty$. It follows from $\theta(0) = \theta_0$ that $C = \theta_0$.

  Thus, the temperature excess is $\theta(x) = \theta_0 e^{-mx}$.

- $\theta'(L) = 0$ (neglible heat loss at the fin tip, "adiabatic fin tip")

  This can be a more reasonable assumption than the infinitely long fin. Note that the total heat transfer from the fin is proportional to its surface area. If the surface area at the fin tip is a negligible fraction of the total surface area, then it is reasonable to assume that $\theta'(L) = 0$.

  In this case, the temperature excess is $\theta(x) = \theta_0 \frac{\cosh(m(L-x))}{\cosh(mL)}$.

  Check! Instead of computing this from scratch (do that as well, later!), check that this indeed solves the DE as well as the boundary conditions $\theta(0) = \theta_0$ and $\theta'(L) = 0$. This should be a rather quick check!

- $\theta(L) = \theta_L$ (specified temperature at fin tip)

  In this case, the temperature excess is $\theta(x) = \frac{\theta_L \sinh(mx) + \theta_0 \sinh(m(L-x))}{\sinh(mL)}$.

  Check! Again, check that this indeed solves the DE as well as the boundary conditions $\theta(0) = \theta_0$ and $\theta(L) = \theta_L$. Once more, this should be a quick and pleasant check.

Armin Straub
straub@southalabama.edu