Review. $\mathcal{L}(u_a(t)f(t-a)) = e^{-as}F(s)$

Here, $u_a(t)f(t-a)$ is $f(t)$ delayed by a.

In particular. $\mathcal{L}(u_a(t)) = \frac{e^{-as}}{s}$ (here, we use $f(t)=1$ and $F(s)=\frac{1}{s}$).

Example 158. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+3}\right)$.

Solution. Since $\frac{e^{-2s}}{s+3}$ features an exponential, we need to use the entry $\mathcal{L}(u_a(t)f(t-a)) = e^{-as}F(s)$.
In our case, $a=2$ and $F(s)=\frac{1}{s+3}$. It follows that $f(t)=e^{-3t}$.
Hence, $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+3}\right) = u_a(t)f(t-a) = u_2(t)e^{-3(t-2)}$.

Comment. Note that this is one of the terms in our solution $Y(s)$ in Example 156 (because $s^2+5s+6=(s+2)(s+3)$). Can you determine the full inverse Laplace transform of $Y(s)$?

In general. Likewise, we have $\mathcal{L}^{-1}\left(\frac{e^{-as}}{s-r}\right) = u_a(t)e^{r(t-a)}$ (namely, e^{rt} delayed by a).

Example 159. Solve the IVP $y''+3y'+2y=f(t)$, $y(0)=y'(0)=0$ with $f(t)=\begin{cases}1, & 3\leqslant t<4,\\ 0, & \text{otherwise.}\end{cases}$

Solution. First, we write $f(t)=u_3(t)-u_4(t)$. We can now take the Laplace transform of the DE to get

$$s^2Y(s)-sy(0)-y'(0)+3(sY(s)-y(0))+2Y(s)=\frac{e^{-3s}}{s}-\frac{e^{-4s}}{s}=(e^{-3s}-e^{-4s})\frac{1}{s}.$$

Using that $s^2+3s+2=(s+1)(s+2)$, we find

$$Y(s)=(e^{-3s}-e^{-4s})\frac{1}{s(s+1)(s+2)}=(e^{-3s}-e^{-4s})\left[\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}\right],$$

where A,B,C are determined by partial fractions (we compute the values below). Taking the inverse Laplace transform of each of the six terms in this product, as in Example 158, we find

$$y(t)=A(u_3(t)-u_4(t))+B(u_3(t)e^{-(t-3)}-u_4(t)e^{-(t-4)})+C(u_3(t)e^{-2(t-3)}-u_4(t)e^{-2(t-4)}).$$

If preferred, we can express this as $y(t)=\begin{cases}0, & \text{if } t<3,\\ A+Be^{-(t-3)}+Ce^{-2(t-3)}, & \text{if } 3\leqslant t<4,\\ B(e^{-(t-3)}-e^{-(t-4)})+C(e^{-2(t-3)}-e^{-2(t-4)}), & \text{if } t\geqslant 4.\end{cases}$

Finally, $A=\left.\frac{1}{(s+1)(s+2)}\right|_{s=0}=\frac{1}{2}$, $B=\left.\frac{1}{s(s+2)}\right|_{s=-1}=-1$, $C=\left.\frac{1}{s(s+1)}\right|_{s=-2}=\frac{1}{2}$.

Comment. Check that these values make $y(t)$ a continuous function (as it should be for physical reasons).

Advanced comment. A close relative to the Laplace transform is the **Fourier transform**:

$$F(\omega)=\int_{-\infty}^{\infty}e^{-i\omega t}f(t)\mathrm{d}t$$

Start with the Laplace transform and note that $s=\sigma+i\omega$ can be complex. If we focus on the purely imaginary case $\sigma=0$, and if $f(t)=0$ for $t<0$, then it turns into the Fourier transform.

We focused on the Laplace transform because it works particularly well for solving DEs. On the other hand, the Fourier transform is only defined if $f(t)$ decays sufficiently but works well for decomposing signals into their constituent frequencies.

Advanced. You may have also seen **Fourier series** which work for functions on a bounded interval $[-L,L]$ (or, equivalently, $2L$-periodic functions), in which case only a single frequency and its multiples appear, whereas the Fourier transform works for functions on the full real line (in a way, it is the limiting case $L\to\infty$).

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Further entries in the Laplace transform table

Finally, we expand our table of Laplace transforms to the following:

$f(t)$	$F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
e^{at}	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{at}f(t)$	$F(s-a)$
$tf(t)$	$-F'(s)$
$u_a(t)f(t-a)$	$e^{-as}F(s)$

Example 160. (new entry) We add the following to our table of Laplace transforms:

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)\mathrm{d}t = \int_0^\infty e^{-(s-a)t}f(t)\mathrm{d}t = F(s-a)$$

Example 161. (new entry) We also add the following to our table of Laplace transforms:

$$\mathcal{L}(tf(t)) = \int_0^\infty e^{-st}tf(t)\mathrm{d}t = \int_0^\infty -\frac{\mathrm{d}}{\mathrm{d}s}e^{-st}f(t)\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st}f(t)\mathrm{d}t = -F'(s)$$

In particular,

$$\begin{aligned} \mathcal{L}(t) &= \mathcal{L}(t\cdot 1) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s} = \frac{1}{s^2} \\ \mathcal{L}(t^2) &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s^2} = \frac{2}{s^3} \\ &\vdots \\ \mathcal{L}(t^n) &= \frac{n!}{s^{n+1}}. \end{aligned}$$

Example 162. Determine the Laplace transform $\mathcal{L}((t-3)e^{2t})$.

Solution. $\mathcal{L}((t-3)e^{2t}) = \mathcal{L}(te^{2t}) - 3\mathcal{L}(e^{2t}) = \frac{1}{(s-2)^2} - \frac{3}{s-2}$

Here, we combined $\mathcal{L}(tf(t)) = -F'(s)$ with $\mathcal{L}(e^{2t}) = \frac{1}{s-2}$ to get $\mathcal{L}(te^{2t}) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s-2} = \frac{1}{(s-2)^2}$.

Alternative. Combine $\mathcal{L}(t-3) = \frac{1}{s^2} - \frac{3}{s}$ and $\mathcal{L}(f(t)e^{2t}) = F(s-2)$ to again get $\mathcal{L}((t-3)e^{2t}) = \frac{1}{(s-2)^2} - \frac{3}{s-2}$.

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