Example 150. (review) Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right)$.

Solution. Note that $s^2-s-6=(s-3)(s+2)$. We use **partial fractions** to write $-\frac{6s-23}{(s-3)(s+2)}=\frac{A}{s-3}+\frac{B}{s+2}$. We find the coefficients (see brief review below) as

$$A=-\frac{6s-23}{s+2}\bigg|_{s=3}=1,\quad B=-\frac{6s-23}{s-3}\bigg|_{s=-2}=-7.$$

Hence $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right)=\mathcal{L}^{-1}\left(\frac{1}{s-3}-\frac{7}{s+2}\right)=\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)-7\mathcal{L}^{-1}\left(\frac{7}{s+2}\right)=e^{3t}-7e^{-2t}$.

Review. In order to find A, we multiply $-\frac{6s-23}{(s-3)(s+2)}=\frac{A}{s-3}+\frac{B}{s+2}$ by $s-3$ to get $-\frac{6s-23}{s+2}=A+\frac{B(s-3)}{s+2}$. We then set $s=3$ to find A as above.

Comment. Compare with Example 139 where we considered the same functions.

Example 151. Consider the IVP $y''-3y'+y=2e^{5t}$, $y(0)=-1$, $y'(0)=4$.

Determine the Laplace transform of the unique solution.

Solution. The DE $y''-3y'+y=2e^{5t}$ (plus initial conditions!) transforms into

$$s^2Y-sy(0)-y'(0)-3(sY-y(0))+Y=(s^2-3s+1)Y+(s-7)=\frac{2}{s-5}.$$

Accordingly, $Y(s)=\frac{1}{s^2-3s+1}\left[\frac{2}{s-5}-s+7\right]$ is the Laplace transform of the unique solution to the IVP.

Comment. The characteristic roots are $(3\pm\sqrt{5})/2$. As a result, the solution $y(t)$ will be rather unpleasant to write down by hand, with coefficients that are not rational numbers. By contrast, the above Laplace transform can be expressed without irrational numbers.

Comment. Depending on what we intend to do with the solution, we might not even need $y(t)$ but might instead be able to extract what we want from its Laplace transform $Y(s)$.