Notes for Lecture 1

Review: Computing derivatives

Given a function $y(x)$, we learned in Calculus I that its **derivative**

$$y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

(where $\Delta y = y(x + \Delta x) - y(x)$) has the following two important characterizations:

- $y'(x)$ is the **slope of the tangent line** of the graph of $y(x)$ at x, and
- $y'(x)$ is the **rate of change** of $y(x)$ at x.

Comment. Derivatives were introduced in the late 1600s by Newton and Leibniz who later each claimed priority in laying the foundations for calculus. Certainly both of them contributed mightily to those foundations.

Moreover, we learned simple rules to compute the derivative of functions:

- **(sum rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = f'(x) + g'(x)$
- **(product rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
- **(chain rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x)$

  Comment. If we write $t = g(x)$ and $y = f(t)$, then the chain rule takes the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x}$.
  In other words, the chain rule expresses the fact that we can treat $\frac{\mathrm{d}y}{\mathrm{d}x}$ (which initially is just a notation for $y'(x)$) as an honest fraction.

- **(basic functions)** $\frac{\mathrm{d}}{\mathrm{d}x} x^r = r x^{r-1}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$, $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} \sin(x) = \cos(x)$, $\frac{\mathrm{d}}{\mathrm{d}x} \cos(x) = -\sin(x)$

These rules are enough to compute the derivative of any function that we can build from the basic functions using algebraic operations and composition. On the other hand, as you probably recall from Calculus II, reversing the operation of differentiation (i.e. computing antiderivatives) is much more difficult.

In particular, there exist simple functions (such as e^{x^2}) whose antiderivative cannot be expressed in terms of the basic functions above.

Example 1. Derive the **quotient rule** from the rules above.

Solution. We write $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$ and apply the product rule to get

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x) \frac{1}{g(x)} + f(x) \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{g(x)}.$$

By the chain rule combined with $\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{x} = -\frac{1}{x^2}$, we have $\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{g(x)} = -\frac{1}{g(x)^2} g'(x)$. Using this in the previous formula,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x) \frac{1}{g(x)} - f(x) \frac{1}{g(x)^2} g'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2}.$$

Putting the final two fractions on a common denominator, we obtain the familiar quotient rule

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Armin Straub
straub@southalabama.edu

Example 2. Compute the following derivatives:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(5x^3+7x^2+2)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(5x^3+7x^2+2)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}(x^3+2x)\sin(5x^3+7x^2+2)$

Solution.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(5x^3+7x^2+2)=15x^2+14x$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(5x^3+7x^2+2)=(15x^2+14x)\cos(5x^3+7x^2+2)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}(x^3+2x)\sin(5x^3+7x^2+2)$
$=(3x^2+2)\sin(5x^3+7x^2+2)+(x^3+2x)(15x^2+14x)\cos(5x^3+7x^2+2)$

First examples of differential equations

Example 3. If $y(x)=e^{x^2}$ then $y'(x)=2xe^{x^2}=2xy(x)$ or, for short, $y'=2xy$.

Accordingly, we say that $y(x)=e^{x^2}$ is a **solution** to the **differential equation** (DE) $y'=2xy$.

Comment. Note that $y(x)=e^{x^2}$ also is a solution to the differential equation $y'=2xe^{x^2}$. Because this DE only involves y' but not y, we can solve it by computing an antiderivative of $2xe^{x^2}$.

Example 4.

(a) By computing its derivative, determine a DE solved by $y(x)=\sin(3x)$.

(b) By computing its second derivative, determine another DE solved by $y(x)=\sin(3x)$.

Solution.

(a) $y'(x)=3\cos(3x)=3\sqrt{1-(\sin(3x))^2}=3\sqrt{1-y(x)^2}$ (for x close to 0).

[Here we used that $\cos(x)^2+\sin(x)^2=1$, which implies that $\cos(x)=\sqrt{1-\sin(x)^2}$.]

Hence, $y(x)=\sin(3x)$ solves the differential equation $y'=3\sqrt{1-y^2}$.

Comment. In the above, we restrict x to $\left(-\frac{\pi}{6},\frac{\pi}{6}\right)$ so that $\cos(3x)>0$. Less precisely, we can say that x is close to 0. (It is a common feature of DEs that we work with values of x close to a certain initial value.)

Comment. Another possible DE would simply be $y'=3\cos(3x)$. However, that is not an "interesting" choice. In particular, this DE could be simply solved by computing an antiderivative. More next time!

(b) $y''(x)=-9\sin(3x)=-9y(x)$.

Thus, $y(x)=\sin(3x)$ also solves the differential equation $y''=-9y$.

Comment. Proceeding the same way, we can check that $y(x)=\cos(3x)$ also solves the DE $y''=-9y$. In fact, so does any linear combination $y(x)=A\cos(x)+B\sin(x)$ (where A and B are constants).

If the highest derivative appearing in a DE is an rth derivative, we say that the DE has **order** r.

For instance. The DE $y'=3\sqrt{1-y^2}$ has order 1 (such DEs are also called first order DEs).
On the other hand, the DE $y''=-9y$ has order 2 (such DEs are also called second order DEs).

Example 5. (cont'd) Determine several (random) DEs that $y(x)=\sin(3x)$ solves.

Solution.

(a) We compute $y'(x)=3\cos(3x)$.

Accordingly, $y(x)=\sin(3x)$ solves the DE $y'=3\cos(3x)$.

Comment. Note that there are further solutions to this DE: the **general solution** is $\int 3\cos(3x)\mathrm{d}x = \sin(3x)+C$ where C is any constant. We say that $y(x)=\sin(3x)+C$ is a **one-parameter family** of solutions to the DE. C is called a **degree of freedom**.

(b) As last time, we note that $\cos(3x)=3\sqrt{1-(\sin(3x))^2}=3\sqrt{1-y(x)^2}$ (for x close to 0).

Hence, $y(x)=\sin(3x)$ solves the differential equation $y'=3\sqrt{1-y^2}$ (for x close to 0).

(c) We compute $y''(x)=-9\sin(3x)$.

Accordingly, $y(x)=\sin(3x)$ solves the DE $y''=-9\sin(3x)$.

Comment. Once more this DE is easy (because it only involves y'' but not y or y'). Hence, we can find the general solution by simply taking two antiderivatives:

$$y(x)=\iint -9\sin(3x)\mathrm{d}x\,\mathrm{d}x=\int (3\cos(3x)+C)\mathrm{d}x=\sin(3x)+Cx+D.$$

Here it is important that we give the second constant of integration a name different from the first. That way, we see that the general solution has 2 degrees of freedom. This matches the fact that the order of the DE is 2.

Important comment. This is no coincidence. In general, we expect a DE of order r to have a general solution with r parameters.

(d) $y(x)=\sin(3x)$ also solves the DE $y''=-9y$.

Comment. This is again a DE of order 2. Therefore the general solution should have 2 degrees of freedom. Later we will learn to solve such DEs. For now, we can verify that $y(x)=A\sin(3x)+B\cos(3x)$ is a solution for any constants A and B.

Homework. Check that $y(x)=\sin(3x)+C$ does not solve the DE $y''=-9y$.

Example 6. Consider the DE $e^y y'=1$.

(a) Is $y(x)=\ln(x)$ a solution to the DE?

(b) Is $y(x)=\ln(x)+C$ a solution to the DE?

(c) Is $y(x)=\ln(x+C)$ a solution to the DE?

Solution.

(a) Since $y'(x)=\frac{1}{x}$ and $e^{y(x)}=e^{\ln(x)}=x$, we have $e^y y'=x\cdot\frac{1}{x}\overset{\checkmark}{=}1$.

Hence, $y(x)=\ln(x)$ is a solution to the given DE.

(b) Since $y'(x)=\frac{1}{x}$ and $e^{y(x)}=e^{\ln(x)+C}=x e^C$, we have $e^y y'=x e^C\cdot\frac{1}{x}=e^C$. Thus the DE is satisfied only if $e^C=1$ which only happens if $C=0$ (which is the case in the first part).

Hence, $y(x)=\ln(x)+C$ is not a solution to the given DE except if $C=0$.

(c) Since $y'(x)=\frac{1}{x+C}$ and $e^{y(x)}=e^{\ln(x+C)}=x+C$, we have $e^y y'=(x+C)\cdot\frac{1}{x+C}\overset{\checkmark}{=}1$.

Hence, $y(x)=\ln(x+C)$ is indeed a one-parameter family of solutions to the given DE.

Example 7. Solve the DE $y' = x^2 + x$.

Solution. Note that the DE simply asks for a function $y(x)$ with a specific derivative (in particular, the right-hand side does not involve $y(x)$). In other words, the desired $y(x)$ is an **antiderivative** of $x^2 + x$. We know from Calculus II that we can find antiderivatives by integrating:

$$y(x) = \int (x^2 + x)\mathrm{d}x = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Moreover, we know from Calculus II that there are no other solutions. In other words, we found the **general solution** to the DE.

To single out a **particular solution**, we need to specify additional conditions (typically one condition per parameter in the general solution). For instance, it is common to impose **initial conditions** such as $y(1) = 2$. A DE together with an initial condition is called an **initial value problem** (IVP).

Example 8. Solve the IVP $y' = x^2 + x$ with $y(1) = 2$.

Solution. From the previous example, we know that $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$.

Since $y(1) = \frac{1}{3} + \frac{1}{2} + C = \frac{5}{6} + C \overset{!}{=} 2$, we find $C = 2 - \frac{5}{6} = \frac{7}{6}$.

Hence, $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{7}{6}$ is the (unique) solution of the IVP.

Example 9. (homework) Solve the DE $y'' = x^2 + x$.

Solution. We now take two antiderivatives of $x^2 + x$ to get

$$y(x) = \iint (x^2 + x)\mathrm{d}x\mathrm{d}x = \int \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + C\right)\mathrm{d}x = \frac{1}{12}x^4 + \frac{1}{6}x^3 + Cx + D,$$

where it is important that we give the second constant of integration a name different from the first.

Important comment. Again, this is the general solution to the DE. The DE is of order 2 and, as expected, the general solution has 2 parameters.

Example 10. (warmup) Consider the DE $y'' = y' + 6y$.

(a) Is $y(x) = e^{2x}$ a solution?

(b) Is $y(x) = e^{3x}$ a solution?

Solution.

(a) $y' = 2e^{2x}$ and $y'' = 4e^{2x}$.
Since $y' + 6y = 8e^{2x}$ is different from $y'' = 4e^{2x}$, we conclude that $y(x) = e^{2x}$ is not a solution.

(b) $y' = 3e^{3x}$ and $y'' = 9e^{3x}$.
Since $y' + 6y = 9e^{3x}$ is equal to $y'' = 9e^{3x}$, we conclude that $y(x) = e^{3x}$ is a solution of the DE.

Example 11. (cont'd) Consider the DE $y'' = y' + 6y$. For which r is e^{rx} a solution?

Solution. If $y(x) = e^{rx}$, then $y'(x) = re^{rx}$ and $y''(x) = r^2 e^{rx}$.
Plugging $y(x) = e^{rx}$ into the DE, we get $r^2 e^{rx} = re^{rx} + 6e^{rx}$ which simplifies to $r^2 = r + 6$.
This has the two solutions $r = -2$, $r = 3$. Hence e^{-2x} and e^{3x} are solutions of the DE.

In fact, we check that $Ae^{-2x} + Be^{3x}$ is a **two-parameter family** of solutions to the DE.

Important comment. It is no coincidence that the order of the DE is 2, whereas the previous example has order 1. In general, we expect a DE of order r to have a solution with r parameters.

Example 12. (extra)

Comment. In this example, we use $x(t)$ instead of $y(x)$ for the function described by the differential equation. In general, of course, any choice of variable names is possible. If we write something like x' or y' it needs to be clear from the context with respect to which variable that derivative is meant (such as $x' = \frac{\mathrm{d}}{\mathrm{d}t} x(t)$).

(a) Verify that $x(t) = \frac{1}{c - kt}$ is a one-parameter family of solutions to the DE $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$.

(b) Solve the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$, $x(0) = 2$.

(c) Solve the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$, $x(0) = 0$.

Solution.

(a) We compute that $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{(c - kt)^2} \cdot (-k) = \frac{k}{(c - kt)^2}$.

On the other hand, $kx^2 = k\left(\frac{1}{c - kt}\right)^2 = \frac{k}{(c - kt)^2}$ as well. Thus, indeed, $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$.

(b) We start with $x(t) = \frac{1}{c - kt}$ (which we know solves the DE for any value of c) and seek to choose c so that $x(0) = 2$.
Since $x(0) = \left[\frac{1}{c - kt}\right]_{t=0} = \frac{1}{c} \overset{!}{=} 2$, we find $c = \frac{1}{2}$.
Hence, the IVP has the (unique) solution $x(t) = \frac{1}{1/2 - kt}$.

(c) Proceeding as in the previous part, we now arrive at the impossible equation $\frac{1}{c} \overset{!}{=} 0$.

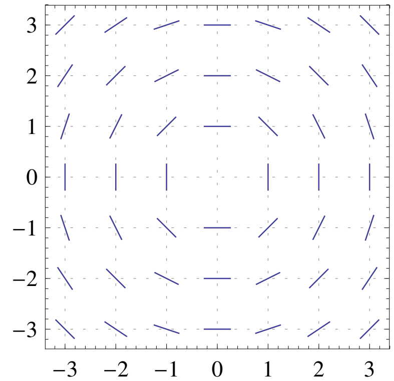
However, this suggests that we should consider taking $c \to \infty$ in $x(t) = \frac{1}{c - kt}$, which results in $x(t) = 0$.

Indeed, it is easy to verify (make sure you know what this entails!) that $x(t) = 0$ solves the IVP.

Armin Straub
straub@southalabama.edu

Slope fields, or sketching solutions to DEs

Example 13. Consider the DE $y' = -x/y$.



Let's pick a point, say, $(1,2)$. If a solution $y(x)$ is passing through that point, then its slope has to be $y' = -1/2$. We therefore draw a small line through the point $(1,2)$ with slope $-1/2$. Continuing in this fashion for several other points, we obtain the **slope field** on the right.

With just a little bit of imagination, we can now anticipate the solutions to look like (half)circles around the origin. Let us check whether $y(x) = \sqrt{r^2 - x^2}$ might indeed be a solution!

$y'(x) = \frac{1}{2}\frac{-2x}{\sqrt{r^2 - x^2}} = -x/y(x)$. So, yes, we actually found solutions!

Solving DEs: Separation of variables

Example 14. Solve the DE $y' = -\frac{x}{y}$.

Solution. Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

Separate the variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$ (in particular, we are multiplying both sides by $\mathrm{d}x$).

Integrating both sides, we get $\int y\,\mathrm{d}y = \int -x\,\mathrm{d}x$.

Computing both integrals results in $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$ (we combine the two constants of integration into one).

Hence $x^2 + y^2 = D$ (with $D = 2C$).

This is an **implicit form** of the solutions to the DE. We can make it explicit by solving for y. Doing so, we find $y(x) = \pm\sqrt{D - x^2}$ (choosing $+$ gives us the upper half of a circle, while the negative sign gives us the lower half).

Comment. The step above where we break $\frac{\mathrm{d}y}{\mathrm{d}x}$ apart and then integrate may sound sketchy!

However, keep in mind that, after we find a solution $y(x)$, even if by sketchy means, we can (and should!) verify that $y(x)$ is indeed a solution by plugging into the DE. We actually already did that in the previous example!

Solving DEs: Separation of variables, cont'd

In general, **separation of variables** solves $y' = g(x)h(y)$ by writing the DE as $\frac{1}{h(y)}\mathrm{d}y = g(x)\mathrm{d}x$.

Note that $\frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$ is indeed equivalent to $\int \frac{1}{h(y)}\mathrm{d}y = \int g(x)\mathrm{d}x + C$. Why?! (Apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to the integrals...)

Example 15. Solve the IVP $y' = -\frac{x}{y}$, $y(0) = -3$.

Comment. Instead of using what we found earlier in Example 14, we start from scratch to better illustrate the solution process (and how we can use the initial condition right away to determine the value of the constant of integration).

Solution. We separate variables to get $y\mathrm{d}y = -x\mathrm{d}x$.

Integrating gives $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$, and we use $y(0) = -3$ to find $\frac{1}{2}(-3)^2 = 0 + C$ so that $C = \frac{9}{2}$.

Hence, $x^2 + y^2 = 9$ is an **implicit** form of the solution.

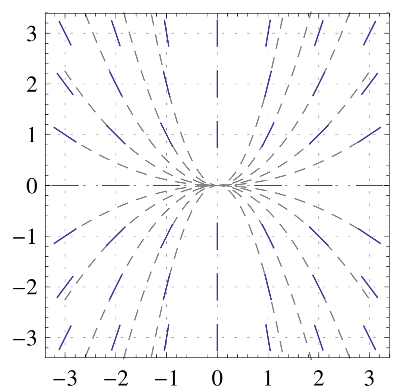
Solving for y, we get $y = -\sqrt{9 - x^2}$ (note that we have to choose the negative sign so that $y(0) = -3$).

Comment. Note that our solution is a **local solution**, meaning that it is valid (and solves the DE) locally around $x = 0$ (from the initial condition). However, it is not a **global solution** because it doesn't make sense outside of x in the interval $[-3, 3]$.

Example 16. Consider the DE $xy' = 2y$.

Sketch its slope field.

Challenge. Try to guess solutions $y(x)$ from the slope field.



Solution. For instance, to find the slope at the point $(3, 1)$, we plug $x = 3$, $y = 1$ into the DE to get $3y' = 2$. Hence, the slope is $y' = 2/3$.

The resulting slope field is sketched on the right.

Solution of the challenge. Trace out the solution through $(1, 1)$ (and then some other points). Their shape looks like a parabola, so that we might guess that $y(x) = Cx^2$ solves the DE. Check that this is indeed the case by plugging into the DE!

Example 17. Solve the IVP $xy' = 2y$, $y(1) = 2$.

Solution. Rewrite the DE as $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x}$.

Then multiply both sides with $\mathrm{d}x$ and integrate both of them to get $\int \frac{1}{y}\mathrm{d}y = \int \frac{2}{x}\mathrm{d}x$.

Hence, $\ln|y| = 2\ln|x| + C$.

The initial condition $y(1) = 2$ tells us that, at least locally, $x > 0$ and $y > 0$. Thus $\ln(y) = 2\ln(x) + C$.

Moreover, plugging in $x = 1$ and $y = 2$, we find $C = \ln(2)$.

Solving $\ln(y) = 2\ln(x) + \ln(2)$ for y, we find $y = e^{2\ln(x) + \ln(2)} = 2x^2$.

Comment. When solving a DE or IVP, we can generally only expect to find a **local solution**, meaning that our solution might only be valid in a small interval around the initial condition (here, we can only expect $y(x)$ to be a solution for all x in an interval around 1; especially since we assumed $x > 0$ in our solution). However, we can check (do it!) that the solution $y = 2x^2$ is actually a **global solution** (meaning that it is a solution for all x, not just locally around 1).

Armin Straub
straub@southalabama.edu

Let's solve the same differential equation with a different choice of initial condition:

Example 18. Solve the IVP $xy' = 2y$, $y(1) = -1$.

Solution. Again, we rewrite the DE as $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x}$, multiply both sides with $\mathrm{d}x$, and integrate to get $\int \frac{1}{y}\mathrm{d}y = \int \frac{2}{x}\mathrm{d}x$.

Hence, $\ln|y| = 2\ln|x| + C$. The initial condition $y(1) = -1$ tells us that, at least locally, $x > 0$ and $y < 0$ (note that this means $|y| = -y$). Thus $\ln(-y) = 2\ln(x) + C$.

Moreover, plugging in $x = 1$ and $y = -1$, we find $C = 0$.

Solving $\ln(-y) = 2\ln(x)$ for y, we find $y = -e^{2\ln(x)} = -x^2$. We easily verify that this is indeed a global solution.

Example 19. $y' = x + y$ is a DE for which the variables cannot be separated.

No worries, very soon we will have several tools to solve this DE as well.

Armin Straub
straub@southalabama.edu

Example 20. (homework) Consider the DE $x^2y' = 1 + xy^3$. Suppose that $y(x)$ is a solution passing through the point $(2, 1)$.

(a) Determine $y'(2)$.

(b) Determine the tangent line of $y(x)$ at $(2, 1)$.

(c) Determine $y''(2)$.

Comment. Note that this DE is not separable.

Solution.

(a) At the point $(2, 1)$ we have $x = 2$ and $y = 1$. Plugging these values into the differential equation, we get $4y' = 1 + 2 \cdot 1^3 = 3$ which we can solve for y' to find $y' = \frac{3}{4}$.

Since y' is short for $y'(x) = y'(2)$, we have found $y'(2) = \frac{3}{4}$.

(b) The tangent line is the line through $(2, 1)$ with slope $\frac{3}{4}$ (computed in the previous part).

From this information, we can immediately write down its equation in the form $y = \frac{3}{4}(x - 2) + 1$.

(c) To get our hands on $y''(2)$, we can differentiate (with respect to x) both sides of $x^2y' = 1 + xy^3$. Applying the product rule, we have $\frac{\mathrm{d}}{\mathrm{d}x}x^2y'(x) = 2xy'(x) + x^2y''(x) = 2xy' + x^2y''$ as well as $\frac{\mathrm{d}}{\mathrm{d}x}(1 + xy(x)^3) = y(x)^3 + x \cdot 3y(x)^2 \cdot y'(x) = y^3 + 3xy^2y'$.
Thus $2xy' + x^2y'' = y^3 + 3xy^2y'$. To find $y''(2)$, we plug in $x = 2$, $y = 1$, $y' = \frac{3}{4}$.

This results in $2 \cdot 2 \cdot \frac{3}{4} + 4y'' = 1 + 3 \cdot 2 \cdot 1 \cdot \frac{3}{4}$ or $3 + 4y'' = \frac{11}{2}$. It follows that $y'' = \frac{1}{4} \cdot \frac{5}{2} = \frac{5}{8}$.

Comment. Alternatively, we can rewrite the DE as $y' = \frac{1}{x^2} + \frac{1}{x}y^3$ and then differentiate. Do it!

Comment. Do you recall from Calculus what it means visually to have $y'' = \frac{5}{8}$?

[Since $y'' > 0$ it means that our function is concave up at $(2, 1)$. As such, its graph will lie above the tangent line.]

Comment. Note that we could continue and likewise find $y'''(2)$ or higher derivatives at $x = 2$. This is the starting point for the power series method typically discussed in Differential Equations II.

ODEs vs PDEs

Important. Note that we are working with functions $y(x)$ of a single variable. This allows us to write simply y' for $\frac{\mathrm{d}}{\mathrm{d}x}y(x)$ without risk of confusion.

Of course, we may use different variables such as $x(t)$ and $x' = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, as long as this is clear from the context.

Differential equations that involve only derivatives with respect to a single variable are known as **ordinary differential equations** (ODEs).
On the other hand, differential equations that involve derivatives with respect to several variables are referred to as **partial differential equations** (PDEs).

Example 21. The DE

$$\left(\frac{\partial}{\partial x}\right)^2 u(x, y) + \left(\frac{\partial}{\partial y}\right)^2 u(x, y) = 0,$$

often abbreviated as $u_{xx} + u_{yy} = 0$, is a partial differential equation in two variables.

This particular PDE is known as **Laplace's equation** and describes, for instance, steady-state heat distributions.
https://en.wikipedia.org/wiki/Laplace%27s_equation

This and other fundamental PDEs will be discussed in Differential Equations II.

Armin Straub
straub@southalabama.edu

Existence and uniqueness of solutions

The following is a very general result that allows us to guarantee that "nice" IVPs must have a solution and that this solution is unique.

Comment. Note that any first-order DE can be written as $g(y', y, x) = 0$ where g is some function of three variables. Assuming that g is reasonable, we can solve for y' and rewrite such a DE as $y' = f(x, y)$ (for some, possibly complicated, function f).

Comment. To be precise, a solution to the IVP $y' = f(x, y)$, $y(a) = b$ is a function $y(x)$, defined on an interval I containing a, such that $y'(x) = f(x, y(x))$ for all $x \in I$ and $y(a) = b$.

Theorem 22. (existence and uniqueness) Consider the IVP $y' = f(x, y)$, $y(a) = b$.
If both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous [in a rectangle] around (a, b), then the IVP has a unique solution in some interval $x \in (a - \delta, a + \delta)$ where $\delta > 0$.

Comment. The interval around a might be very small. In other words, the δ in the theorem could be very small.

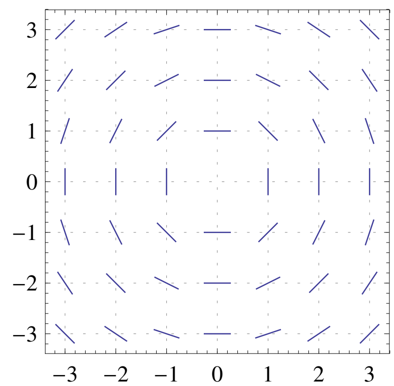
Comment. Note that the theorem makes two important assertions. First, it says that there exists a **local solution**. Second, it says that this solution is unique. These two parts of the theorem are famous results usually attributed to Peano (existence) and Picard–Lindelöf (uniqueness).

Advanced comment. The condition about $\frac{\partial}{\partial y} f(x, y)$ is a bit technical (and not optimal). If we drop this condition, we still get existence but, in general, no longer uniqueness.

Advanced comment. The interval in which the solution is unique could be smaller than the interval in which it exists. In other words, it is possible that, away from the initial condition, the solution "forks" into two or more solutions. Note that this does not contradict the theorem because it only guarantees uniqueness on a small interval.

Review. Existence and uniqueness theorem (Theorem 22) for an IVP $y' = f(x, y)$, $y(a) = b$:
If $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous around (a, b) then, locally, the IVP has a unique solution.

Example 23. Consider, again, the IVP $y' = -x/y$, $y(a) = b$. Discuss existence and uniqueness of solutions (without solving).



Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = -x/y$.

We compute that $\frac{\partial}{\partial y} f(x, y) = x/y^2$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $y \neq 0$.

Hence, if $b \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

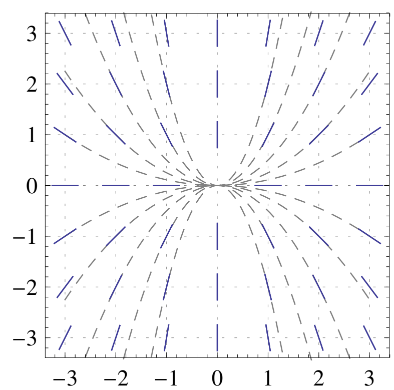
Comment. In Example 14, we found that the DE $y' = -x/y$ is solved by $y(x) = \pm\sqrt{D - x^2}$.

Assume $b > 0$ (things work similarly for $b < 0$). Then $y(x) = \sqrt{D - x^2}$ solves the IVP (we need to choose D so that $y(a) = b$) if we choose $D = a^2 + b^2$. This confirms that there exists a solution. On the other hand, uniqueness means that there can be no other solution to the IVP than this one.

What happens in the case $b = 0$?

Solution. In this case, the existence and uniqueness theorem does not guarantee anything. If $a \neq 0$, then $y(x) = \sqrt{a^2 - x^2}$ and $y(x) = -\sqrt{a^2 - x^2}$ both solve the IVP (so we certainly don't have uniqueness), however only in a weak sense: namely, both of these solutions are not valid locally around $x = a$ but only in an interval of which a is an endpoint (for instance, the IVP $y' = -x/y$, $y(2) = 0$ is solved by $y(x) = \pm\sqrt{4 - x^2}$ but both of these solutions are only valid on the interval $[-2, 2]$ which ends at 2, and neither of these solutions can be extended past 2).

Example 24. Consider, again, the IVP $xy' = 2y$, $y(a) = b$. Discuss existence and uniqueness of solutions (without solving).



Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = 2y/x$.

We compute that $\frac{\partial}{\partial y} f(x, y) = 2/x$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $x \neq 0$.

Hence, if $a \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

What happens in the case $a = 0$?

Solution. In Example 16, we found that the DE $xy' = 2y$ is solved by $y(x) = Cx^2$.

This means that the IVP with $y(0) = 0$ has infinitely many solutions.

On the other hand, the IVP with $y(0) = b$ where $b \neq 0$ has no solutions. (This follows from the fact that there are no solutions to the DE besides $y(x) = Cx^2$. Can you see this by looking at the slope field?)

Armin Straub
straub@southalabama.edu

Example 25. Consider the IVP $y' = ky^2$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = ky^2$. We compute that $\frac{\partial}{\partial y} f(x, y) = 2ky$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y).

Hence, for any initial conditions, the IVP locally has a unique solution by the existence and uniqueness theorem.

Example 26. Solve $y' = ky^2$.

Solution. Separate variables to get $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = k$.

Integrating $\int \frac{1}{y^2}\mathrm{d}y = \int k\mathrm{d}x$, we find $-\frac{1}{y} = kx + C$.

We solve for y to get $y = -\frac{1}{C + kx} = \frac{1}{D - kx}$ (with $D = -C$). That is the solution we verified earlier!

Comment. Note that we did not find the solution $y = 0$ (it was "lost" when we divided by y^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). However, note that we can obtain it from the general solution by letting $D \to \infty$.

Caution. We have to be careful about transforming our DE when using separation of variables: Just as the division by y^2 made us lose a solution, other transformations can add extra solutions which do not solve the original DE. Here is a silly example (silly, because the transformation serves no purpose here) which still illustrates the point. The DE $(y-1)y' = (y-1)ky^2$ has the same solutions as $y' = ky^2$ plus the additional solution $y = 1$ (which does not solve $y' = ky^2$).

Example 27. (extra) Solve the IVP $y' = y^2$, $y(0) = 1$.

Solution. From the previous example with $k = 1$, we know that $y(x) = \frac{1}{D - x}$.

Using $y(0) = 1$, we find that $D = 1$ so that the unique solution to the IVP is $y(x) = \frac{1}{1 - x}$.

Comment. Note that we already concluded the uniqueness from the existence and uniqueness theorem.
On the other hand, note that $y(x) = \frac{1}{1 - x}$ is only valid on $(-\infty, 1)$ and that it cannot be continuously extended past $x = 1$; it is only a local solution.

Example 28. (homework) Consider the IVP $(x - y^2)y' = 3x$, $y(4) = b$. For which choices of b does the existence and uniqueness theorem guarantee a unique (local) solution?

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = 3x/(x - y^2)$. We compute that $\frac{\partial}{\partial y} f(x, y) = 6xy/(x - y^2)^2$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $x - y^2 \neq 0$.

Note that $4 - b^2 \neq 0$ is equivalent to $b \neq \pm 2$.

Hence, if $b \neq \pm 2$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Linear first-order DEs

A **linear differential equation** is one where the function y and its derivatives only show up linearly (i.e. there are no terms such as y^2, $1/y$, $\sin(y)$ or $y \cdot y'$).
As such, the most general linear first-order DE is of the form

$$A(x)y' + B(x)y + C(x) = 0.$$

Comment. Note that any such DE can also be rewritten in the "**standard form**" $y' + P(x)y = Q(x)$ by dividing by $A(x)$ and rearranging. We will use this form when solving linear first-order DEs.

Example 29. (extra "warmup") Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2$.

Solution. (separation of variables) $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, $-\frac{1}{y} = x^2 + C$.

Hence the general solution is $y = \frac{1}{D - x^2}$. [There also is the singular solution $y = 0$.]

Solution. (in other words) Note that $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ can be written as $\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{1}{y}\right] = \frac{\mathrm{d}}{\mathrm{d}x}[x^2]$.
From there it follows that $-\frac{1}{y} = x^2 + C$, as above.

We now use the idea of writing both sides as a derivative to also solve linear DEs that are not separable.

The multiplication by $\frac{1}{y^2}$ will be replaced by multiplication with a so-called **integrating factor**.

Example 30. Solve $y' = x - y$.

Comment. Note that we cannot use separation of variables this time.

Solution. Rewrite the DE as $y' + y = x$.
Next, multiply both sides with e^x (we will see in a little bit how to find this "integrating factor") to get

$$\underbrace{e^x y' + e^x y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^x y]} = x e^x.$$

The "magic" part is that we are able to realize the new left-hand side as a derivative!
We can then integrate both sides to get

$$e^x y = \int x e^x \mathrm{d}x = x e^x - e^x + C.$$

From here it follows that $y = x - 1 + Ce^{-x}$.

Comment. For the final integral, we used that $\int x e^x \mathrm{d}x = x e^x - \int e^x \mathrm{d}x = x e^x - e^x + C$ which follows, for instance, via integration by parts (with $f(x) = x$ and $g'(x) = e^x$ in the formula reviewed below).

Review. The multiplication rule $(fg)' = f'g + fg'$ implies $fg = \int f'g + \int fg'$.
The latter is equivalent to **integration by parts**:

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x$$

Comment. Sometimes, one writes $g'(x)\mathrm{d}x = \mathrm{d}g(x)$.

Armin Straub
straub@southalabama.edu

In general, we can solve any **linear first-order DE** $y' + P(x)y = Q(x)$ in this way.

- We want to multiply with an **integrating factor** $f(x)$ such that the left-hand side of the DE becomes

$$f(x)y' + f(x)P(x)y = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)y].$$

  Since $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)y] = f(x)y' + f'(x)y$, we need $f'(x) = f(x)P(x)$ for that.

- Check that $f(x) = \exp\left(\int P(x)\mathrm{d}x\right)$ has this property.

  Comment. This follows directly from computing the derivative of this $f(x)$ via the chain rule.

  Homework. On the other hand, note that finding f meant solving the DE $f' = P(x)\,f$. This is a separable DE. Solve it by separation of variables to arrive at the above formula for $f(x)$ yourself.

  Just to make sure. There is no difference between $\exp(x)$ and e^x. Here, we prefer the former notation for typographical reasons.

With that integrating factor, we have the following recipe for solving any linear first-order equation:

(solving linear first-order DEs)

(a) Write the DE in the **standard form** $y' + P(x)y = Q(x)$.

(b) Compute the **integrating factor** as $f(x) = \exp\left(\int P(x)\mathrm{d}x\right)$.

[We can choose any constant of integration.]

(c) Multiply the DE from part (a) by $f(x)$ to get

$$\underbrace{f(x)y' + f(x)P(x)y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[f(x)y]} = f(x)Q(x).$$

(d) Integrate both sides to get

$$f(x)y = \int f(x)Q(x)\mathrm{d}x + C.$$

Then solve for y by dividing by $f(x)$.

Comment. For better understanding, we prefer to go through the above steps. On the other hand, we can combine these steps into the following formula for the general solution of $y' + P(x)y = Q(x)$:

$$y = \frac{1}{f(x)}\left(\int f(x)Q(x)\mathrm{d}x + C\right) \quad \text{where } f(x) = e^{\int P(x)\mathrm{d}x}$$

Existence and uniqueness. Note that the solution we construct exists on any interval on which P and Q are continuous (not just on some possibly very small interval). This is better than what the existence and uniqueness theorem (Theorem 22) can guarantee. This is one of the many ways in which linear DEs have particularly nice properties compared to DEs in general.

Example 31. Solve $x^2 y' = 1 - xy + 2x$, $y(1) = 3$.

Solution. This is a linear first-order DE. We can therefore solve it according to the recipe above.

(a) Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ (standard form) with $P(x) = \frac{1}{x}$ and $Q(x) = \frac{1}{x^2} + \frac{2}{x}$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^{\ln x} = x$.

Here, we could write $\ln x$ instead of $\ln|x|$ because the initial condition tells us that $x > 0$, at least locally.
Comment. We can also choose a different constant of integration but that would only complicate things.

(c) Multiply the DE (in standard form) by $f(x) = x$ to get

$$\underbrace{x\frac{\mathrm{d}y}{\mathrm{d}x} + y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[xy]} = \frac{1}{x} + 2.$$

(d) Integrate both sides to get (again, we use that $x > 0$ to avoid having to use $|x|$)

$$xy = \int \left(\frac{1}{x} + 2\right)\mathrm{d}x = \ln x + 2x + C.$$

Using $y(1) = 3$ to find C, we get $1 \cdot 3 = \ln(1) + 2 \cdot 1 + C$ which results in $C = 3 - 2 = 1$.
Hence, the (unique) solution to the IVP is $y = \frac{\ln(x) + 2x + 1}{x}$.

Example 32. (review) Solve $xy' = 2y + 1$, $y(-2) = 0$.

Solution. This is a linear first-order DE.

(a) Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ (standard form) with $P(x) = -\frac{2}{x}$ and $Q(x) = \frac{1}{x}$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^{-2\ln|x|} = e^{-2\ln(-x)} = (-x)^{-2} = \frac{1}{x^2}$.

Here, we used that, at least locally, $x < 0$ (because the initial condition is $x = -2 < 0$) so that $|x| = -x$.

(c) Multiply the DE (in standard form) by $f(x) = \frac{1}{x^2}$ to get

$$\underbrace{\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x^3}y}_{= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x^2}y\right]} = \frac{1}{x^3}.$$

(d) Integrate both sides to get

$$\frac{1}{x^2}y = \int \frac{1}{x^3}\mathrm{d}x = -\frac{1}{2x^2} + C.$$

Hence, the general solution is $y(x) = -\frac{1}{2} + Cx^2$.

Solving $y(-2) = -\frac{1}{2} + 4C = 0$ for C yields $C = \frac{1}{8}$. Thus, the (unique) solution to the IVP is $y(x) = \frac{1}{8}x^2 - \frac{1}{2}$.

Example 33. (extra) Solve $y' = 2y + 3x - 1$, $y(0) = 2$.

Solution. This is a linear first-order DE.

(a) Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ (standard form) with $P(x) = -2$ and $Q(x) = 3x - 1$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^{-2x}$.

(c) Multiply the DE (in standard form) by $f(x) = e^{-2x}$ to get

$$\underbrace{e^{-2x}\frac{\mathrm{d}y}{\mathrm{d}x} - 2e^{-2x}y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^{-2x}y]} = (3x - 1)e^{-2x}.$$

(d) Integrate both sides to get

$$\begin{aligned} e^{-2x}y &= \int (3x-1)e^{-2x}\mathrm{d}x \\ &= 3\int xe^{-2x}\mathrm{d}x - \int e^{-2x}\mathrm{d}x \\ &= 3\left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right) - \left(-\frac{1}{2}e^{-2x}\right) + C \\ &= -\frac{3}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C. \end{aligned}$$

Here, we used that $\int xe^{-2x}\mathrm{d}x = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\mathrm{d}x = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$ (for instance, via integration by parts with $f(x) = x$ and $g'(x) = e^{-2x}$).

Hence, the general solution is $y(x) = -\frac{3}{2}x - \frac{1}{4} + Ce^{2x}$.

Solving $y(0) = -\frac{1}{4} + C = 2$ for C yields $C = \frac{9}{4}$.

In conclusion, the (unique) solution to the IVP is $y(x) = -\frac{3}{2}x - \frac{1}{4} + \frac{9}{4}e^{2x}$.

Armin Straub
straub@southalabama.edu

Example 34. (review) Using substitution, compute $\int \frac{x}{1+x^2}\mathrm{d}x$.

Solution. We substitute $t = 1 + x^2$. In that case, $\mathrm{d}t = 2x\,\mathrm{d}x$.

$\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\int \frac{1}{t}\mathrm{d}t = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln(1+x^2) + C$

Comment. Why were we allowed to drop the absolute value in the logarithm?

Review. On the other hand, recall that $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan(x) + C$.

Example 35. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = (x+y)^2$.

First things first. Is this DE separable? Is it linear? (No to both but make sure that this is clear to you.)
This means that our previous techniques are not sufficient to solve this DE.

Solution. Looking at the right-hand side, we have a feeling that the substitution $u = x + y$ might simplify things.
Then $y = u - x$ and, therefore, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} - 1$.

Using these, the DE translates into $\frac{\mathrm{d}u}{\mathrm{d}x} - 1 = u^2$. This is a separable DE: $\frac{1}{1+u^2}\,\mathrm{d}u = \mathrm{d}x$

After integration, we find $\arctan(u) = x + C$ and, thus, $u = \tan(x + C)$.
The solution of the original DE is $y = u - x = \tan(x + C) - x$.

Example 36. (homework) Consider the DE $x\frac{dy}{dx} = y + y^2 f(x)$.

(a) Substitute $u = \frac{y}{x}$. Is the resulting DE separable or linear?

(b) Substitute $v = \frac{1}{y}$. Is the resulting DE separable or linear?

(c) Solve each of the new DEs.

Solution.

(a) Set $u = \frac{y}{x}$. Then $y = ux$ and, thus, $\frac{dy}{dx} = x\frac{du}{dx} + u$.
Using these, the DE translates into $x\left(x\frac{du}{dx} + u\right) = ux + (ux)^2 f(x)$.
This DE simplifies to $\frac{du}{dx} = u^2 f(x)$. This is a separable DE.

(b) Set $v = \frac{1}{y}$. Then $y = \frac{1}{v}$ and, thus, $\frac{dy}{dx} = -\frac{1}{v^2}\frac{dv}{dx}$.
Using these, the DE translates into $x\left(-\frac{1}{v^2}\frac{dv}{dx}\right) = \frac{1}{v} + \frac{1}{v^2} f(x)$.
This DE simplifies to $x\frac{dv}{dx} = -v - f(x)$. This is a linear DE.

(c) Let us write $F(x)$ for an antiderivative of $f(x)$.

- The DE $\frac{du}{dx} = u^2 f(x)$ from the first part is separable: $u^2 du = f(x)dx$.
  After integration, we find $-\frac{1}{u} = F(x) + C$.
  Since $u = \frac{y}{x}$, this becomes $-\frac{x}{y} = F(x) + C$.
  The general solution of the initial DE therefore is $y = -\frac{x}{F(x) + C}$.

- The DE $x\frac{dv}{dx} = -v - f(x)$ from the second part is linear. We apply our recipe:

  (a) Rewrite the DE as $\frac{dv}{dx} + P(x)v = Q(x)$ with $P(x) = 1/x$ and $Q(x) = -f(x)/x$.

  (b) The integrating factor is $\exp\left(\int P(x)dx\right) = e^{\ln x} = x$.

  Comment. We should make a mental note that we assumed that $x > 0$. In the next step, however, we see that the integrating factor works for all x.

  (c) Multiply the (rewritten) DE by the integrating factor x to get $\underbrace{x\frac{dv}{dx} + v}_{=\frac{d}{dx}[xv]} = -f(x)$.

  (d) Integrate both sides to get $xv = -F(x) + C$.

  Since $v = \frac{1}{y}$, we find $\frac{x}{y} = -F(x) + C$.
  The general solution of the initial DE therefore is $y = -\frac{x}{F(x) - C}$.

Comment. Note that our two approaches led to the same general solution (from the existence and uniqueness theorem, we can see that this must be the case). One of the formulas features $+C$ while the other features $-C$. However, that makes no difference because C is a free parameter (we could have given them different names if we preferred).

Armin Straub
straub@southalabama.edu

Useful substitutions

The previous example illustrates that different substitutions can help to solve a given DE. Choosing the right substitution is difficult in general. The following is a compilation of important cases that are easy to spot and for which the listed substitutions are guaranteed to succeed:

- $y' = F\left(\frac{y}{x}\right)$

  Set $u = \frac{y}{x}$. Then $y = ux$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$. We get $x\frac{\mathrm{d}u}{\mathrm{d}x} + u = F(u)$. This DE is always separable.

  Caution. The DE $y' = F\left(\frac{y}{x}\right)$ is sometimes called a "homogeneous equation". However, we will soon discuss homogeneous linear differential equations, where the label homogeneous means something different (though in both cases, there is a common underlying reason).

- $y' = F(ax + by)$

  Set $u = ax + by$. Then $y = \frac{1}{b}(u - ax)$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{b}\left(\frac{\mathrm{d}u}{\mathrm{d}x} - a\right)$.

  The new DE is $\frac{1}{b}\left(\frac{\mathrm{d}u}{\mathrm{d}x} - a\right) = F(u)$ or, simplified, $\frac{\mathrm{d}u}{\mathrm{d}x} = a + bF(u)$. This DE is always separable.

- $y' = F(x)y + G(x)y^n$ (This is called a **Bernoulli equation**.)

  Set $u = y^{1-n}$. The resulting DE is always linear.

  Details. If $u = y^{1-n}$ then $y = u^{1/(1-n)}$ and, thus, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1-n}u^{n/(1-n)}\frac{\mathrm{d}u}{\mathrm{d}x}$. $\left[\frac{1}{1-n} - 1 = \frac{n}{1-n}\right]$

  The new DE is $\frac{1}{1-n}u^{n/(1-n)}\frac{\mathrm{d}u}{\mathrm{d}x} = F(x)u^{1/(1-n)} + G(x)u^{n/(1-n)}$.

  Dividing both sides by $u^{n/(1-n)}$, the DE simplifies to $\frac{1}{1-n}\frac{\mathrm{d}u}{\mathrm{d}x} = F(x)u + G(x)$ which is a linear DE.

  Comment. The original DE has the trivial solution $y = 0$. Do you see where we lost that solution?

Example 37. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = (2x - 3y)^2 + \frac{2}{3}$, $y(1) = \frac{1}{3}$.

Solution. This is of the form $y' = F(2x - 3y)$ with $F(t) = t^2 + \frac{2}{3}$.

Therefore, as suggested by the above list, we substitute $u = 2x - 3y$.

Then $y = \frac{1}{3}(2x - u)$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}\left(2 - \frac{\mathrm{d}u}{\mathrm{d}x}\right)$.

The new DE is $\frac{1}{3}\left(2 - \frac{\mathrm{d}u}{\mathrm{d}x}\right) = u^2 + \frac{2}{3}$ or, simplified, $\frac{\mathrm{d}u}{\mathrm{d}x} = -3u^2$.

This DE is separable: $u^{-2}\mathrm{d}u = -3\mathrm{d}x$. After integration, $-\frac{1}{u} = -3x + C$.

We conclude that $u = \frac{1}{3x - C}$ and, hence, $y(x) = \frac{1}{3}(2x - u) = \frac{2}{3}x - \frac{1}{3}\frac{1}{3x - C}$.

Solving $y(1) = \frac{2}{3} - \frac{1}{3}\frac{1}{3 - C} = \frac{1}{3}$ for C leads to $C = 2$.

Hence, the unique solution of the IVP is $y(x) = \frac{2}{3}x - \frac{1}{3(3x - 2)}$.

Example 38. Solve $(x - y)\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$.

Solution. Divide the DE by x to get $\left(1 - \frac{y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{y}{x}$. This is a DE of the form $y' = F\left(\frac{y}{x}\right)$.

We therefore substitute $u = \frac{y}{x}$. Then $y = ux$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$.

The resulting DE is $(x - ux)\left(x\frac{\mathrm{d}u}{\mathrm{d}x} + u\right) = x + ux$, which simplifies to $x(1 - u)\frac{\mathrm{d}u}{\mathrm{d}x} = 1 + u^2$.

This DE is separable: $\frac{1 - u}{1 + u^2}\mathrm{d}u = \frac{1}{x}\mathrm{d}x$

Integrating both sides, we find $\arctan(u) - \frac{1}{2}\ln(1 + u^2) = \ln|x| + C$.

Setting $u = y/x$, we get the (general) implicit solution $\arctan(y/x) - \frac{1}{2}\ln(1 + (y/x)^2) = \ln|x| + C$.

Comment. We used $\int \frac{1}{1 + u^2}\mathrm{d}u = \arctan(u) + C$ and $\int \frac{x}{1 + x^2}\mathrm{d}x = \frac{1}{2}\ln(1 + x^2) + C$ when integrating.

See Example 34 where we reviewed these integrals.

Example 39. Solve the IVP $\frac{dy}{dx} = 2y - 3xy^5$, $y(0) = 1$.

Solution. This is an example of a Bernoulli equation (with $n = 5$). We therefore substitute $u = y^{1-n} = y^{-4}$. Accordingly, $y = u^{-1/4}$ and, thus, $\frac{dy}{dx} = -\frac{1}{4}u^{-5/4}\frac{du}{dx}$.

The new DE is $-\frac{1}{4}u^{-5/4}\frac{du}{dx} = 2u^{-1/4} - 3xu^{-5/4}$, which simplifies to $\frac{du}{dx} = -8u + 12x$.

This is a linear first-order DE, which we solve according to our recipe:

(a) Rewrite the DE as $\frac{du}{dx} + P(x)u = Q(x)$ with $P(x) = 8$ and $Q(x) = 12x$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^{8x}$.

(c) Multiply the (rewritten) DE by $f(x) = e^{8x}$ to get

$$\underbrace{e^{8x}\frac{du}{dx} + 8e^{8x}u}_{= \frac{d}{dx}[e^{8x}u]} = 12xe^{8x}.$$

(d) Integrate both sides to get:

$$e^{8x}u = 12\int xe^{8x}dx = 12\left(\frac{1}{8}xe^{8x} - \frac{1}{8^2}e^{8x}\right) + C = \frac{3}{2}xe^{8x} - \frac{3}{16}e^{8x} + C$$

Here we used that $\int xe^{ax}dx = \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax}$. (Integration by parts!)

The general solution of the DE for u therefore is $u = \frac{3}{2}x - \frac{3}{16} + Ce^{-8x}$.

Correspondingly, the general solution of the initial DE is $y = u^{-1/4} = 1/\sqrt[4]{\frac{3}{2}x - \frac{3}{16} + Ce^{-8x}}$.

Using $y(0) = 1$, we find $1 = 1/\sqrt[4]{C - \frac{3}{16}}$ from which we obtain $C = 1 + \frac{3}{16} = \frac{19}{16}$.

The unique solution to the IVP therefore is $y = 1/\sqrt[4]{\frac{3}{2}x - \frac{3}{16} + \frac{19}{16}e^{-8x}}$.

Solving simple 2nd order DEs

We have the following two useful substitutions for certain simple DEs of order 2:

- $F(y'', y', x) = 0$ (2nd order with "y missing")

  Set $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx}$. We get the first-order DE $F\left(\frac{du}{dx}, u, x\right) = 0$.

- $F(y'', y', y) = 0$ (2nd order with "x missing")

  Set $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u$. We get the first-order DE $F\left(u\frac{du}{dy}, u, y\right) = 0$.

Example 40. Solve $y'' = x - y'$.

Solution. We substitute $u = y'$, which results in the first-order DE $u' = x - u$.
This DE is linear and, using our recipe (see below for the details), we can solve it to find $u = x - 1 + Ce^{-x}$.
Since $y' = u$, we conclude that the general solution is $y = \int (x - 1 + Ce^{-x})dx = \frac{1}{2}x^2 - x - Ce^{-x} + D$.

Important comment. This is a DE of order 2. Hence, as expected, the general solution has two free parameter.

Solving the linear DE. To solve $u' = x - u$ (also see Example 30, where we had solved this DE before), we

(a) rewrite the DE as $\frac{du}{dx} + P(x)u = Q(x)$ with $P(x) = 1$ and $Q(x) = x$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^x$.

(c) Multiply the (rewritten) DE by $f(x) = e^x$ to get $\underbrace{e^x\frac{du}{dx} + e^x u}_{= \frac{d}{dx}[e^x u]} = xe^x$.

(d) Integrate both sides to get (using integration by parts): $e^x u = \int xe^x dx = xe^x - e^x + C$

Hence, the general solution of the DE for u is $u = x - 1 + Ce^{-x}$, which is what we used above.

Example 41. (homework) Solve the IVP $y'' = x - y'$, $y(0) = 1$, $y'(0) = 2$.

Solution. As in the previous example, we find that the general solution to the DE is $y(x) = \frac{1}{2}x^2 - x - Ce^{-x} + D$.
Using $y'(x) = x - 1 + Ce^{-x}$ and $y'(0) = 2$, we find that $2 = -1 + C$. Hence, $C = 3$.
Then, using $y(x) = \frac{1}{2}x^2 - x - 3e^{-x} + D$ and $y(0) = 1$, we find $1 = -3 + D$. Hence, $D = 4$.
In conclusion, the unique solution to the IVP is $y(x) = \frac{1}{2}x^2 - x - 3e^{-x} + 4$.

Example 42. (extra) Find the general solution to $y'' = 2yy'$.

Solution. We substitute $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u$.
Therefore, our DE turns into $u\frac{du}{dy} = 2yu$.
Dividing by u, we get $\frac{du}{dy} = 2y$. [Note that we lose the solution $u = 0$, which gives the singular solution $y = C$.]
Hence, $u = y^2 + C$. It remains to solve $y' = y^2 + C$. This is a separable DE.
$\frac{1}{C + y^2}dy = dx$. Let us restrict to $C = D^2 \geqslant 0$ here. (This means we will only find "half" of the solutions.)
$\int \frac{1}{D^2 + y^2}dy = \frac{1}{D^2}\int \frac{1}{1 + (y/D)^2}dy = \frac{1}{D}\arctan(y/D) = x + A$.
Solving for y, we find $y = D\tan(Dx + AD) = D\tan(Dx + B)$. $[B = AD]$

Armin Straub
straub@southalabama.edu

Applications of DEs & Modeling

The exponential model of population growth

If $P(t)$ is the size of a population (eg. of bacteria) at time t, then the rate of change $\frac{\mathrm{d}P}{\mathrm{d}t}$ might, from biological considerations, be (nearly) proportional to $P(t)$.

Why? This might be more clear if we use some (random) numbers. Say, we have a population of $P=100$ and $P'=3$, meaning that the population changes by 3 individuals per unit of time. By how do we expect a population of $P=500$ to change? (Think about it for a moment!)

[Without further information, we would probably expect the population of $P=500$ to change by $5\cdot 3=15$ individuals per unit of time, so that $P'=15$ in that case. This is what it means for P' to be proportional to P. In formulas, it means that P'/P is constant or, equivalently, that $P'=kP$ for a proportionality constant k.]

Comment. "Population" might sound more specific than it is. It could also refer to rather different populations such as amounts of money (finance) or amounts of radioactive material (physics).

For instance, thinking about an amount $P(t)$ of money in a bank account at time t, we would also expect $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the money per time that we gain from receiving interest) to be proportional to $P(t)$.

The corresponding **mathematical model** is described by the DE $\frac{\mathrm{d}P}{\mathrm{d}t}=kP$ where k is the constant of proportionality.

Example 43. Determine all solutions to the DE $\frac{\mathrm{d}P}{\mathrm{d}t}=kP$.

Solution. We easily guess and then verify that $P(t)=Ce^{kt}$ is a solution. (Alternatively, we can find this solution via separation of variables or because this is a linear DE. Do it both ways!)

Moreover, it follows from the existence and uniqueness theorem that there cannot be further solutions. (Alternatively, we can conclude this from our solving process (separation of variables or our approach to linear DEs only lose solutions when we divide by zero and we can consider those cases separately)).

Mathematics therefore tells us that the (only) solutions to this DE are given by $P(t)=Ce^{kt}$ where C is some constant.

Hence, populations satisfying the assumption from biology necessarily exhibit exponential growth.

Example 44. Let $P(t)$ describe the size of a population at time t. Suppose $P(0)=100$ and $P(1)=300$. Under the exponential model of population growth, find $P(t)$.

Solution. $P(t)$ solves the DE $\frac{\mathrm{d}P}{\mathrm{d}t}=kP$ and therefore is of the form $P(t)=Ce^{kt}$.

We now use the two data points to determine both C and k.

$Ce^{k\cdot 0}=C=100$ and $Ce^{k}=100e^{k}=300$. Hence $k=\ln(3)$ and $P(t)=100e^{\ln(3)t}=100\cdot 3^{t}$.

Main challenge of modeling: a model has to be detailed enough to resemble the real world, yet simple enough to allow for mathematical analysis.

Observe that the exponential model of population growth can be written as

$$\frac{P'}{P}=\text{constant}.$$

Thinking purely mathematically (generally not a good idea for modeling!), to extend the model, it might be sensible to replace constant (which we called k above) by the next simplest kind of function, namely a linear function in P. The resulting

Comment. Can you put into words why we replace constant by a function of P rather than a function of t? When would it be appropriate to add a dependence on t?

[A dependence on t would make sense if the "environment" changes over time. Without such a change, we expect that a population (say, of bacteria in our lab) behaves this week just as it would next week. The "law" behind its growth should not depend on t. The resulting differential equations are called **autonomous**.]

The logistic model of population growth

If the population is constrained by resources, then $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ is not a good model. A model to take that into account is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$. This is the **logistic equation**.

M is called the carrying capacity:

- Note that if $P \ll M$ then $1 - \frac{P}{M} \approx 1$ and we are back to the simpler exponential model. This means that the population P will grow (nearly) exponentially if P is much less than the carrying capacity M.
- On the other hand, if $P > M$ then $1 - \frac{P}{M} < 0$ so that (assuming $k > 0$) $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$, which means that the population P is shrinking if it exceeds the carrying capacity M.

Comment. If $P(t)$ is the size of a population, then P'/P can be interpreted as its *per capita growth rate*.
Note that in the exponential model we have that $P'/P = k$ is constant.
On the other hand, in the logistic model we have that $P'/P = k(1 - P/M)$ is a linear function.

Example 45. Solve the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$.

Solution. This is a separable DE: $\frac{1}{P(1 - \frac{P}{M})}\mathrm{d}P = k\mathrm{d}t$.

To integrate the left-hand side, we use partial fractions: $\frac{1}{P(1 - \frac{P}{M})} = \frac{1}{P} + \frac{1/M}{1 - \frac{P}{M}} = \frac{1}{P} - \frac{1}{P - M}$.

After integrating, we obtain $\ln|P| - \ln|P - M| = kt + A$.

Equivalently, $\ln\left|\frac{P}{P - M}\right| = kt + A$ so that $\frac{P}{P - M} = \pm e^{kt + A} = Be^{kt}$ where $B = \pm e^A$.

Solving for P, we conclude that the general solution is

$$P(t) = \frac{BMe^{kt}}{Be^{kt} - 1} = \frac{M}{1 + Ce^{-kt}},$$

where we replaced the free parameter B with $C = -1/B$.

Initial population. Note that the initial population is $P(0) = \frac{M}{1 + C}$. Equivalently, $C = \frac{M}{P(0)} - 1$ which expresses the free parameter C in terms of the initial population.

Comment. Note that $B = \pm e^A$ can be any real number except 0. However, we can easily check that $B = 0$ also gives us a solution to the DE (namely, the trivial solution $P = 0$). This solution was "lost" when we divided by P to separate variables.

Exercise. Note that the logistic equation is a Bernoulli equation. As an alternative to separation of variables, we can therefore solve it by transforming it to a linear DE.

Review of partial fractions. Recall that partial fractions tells us that fractions like $\frac{p(x)}{(x - r_1)(x - r_2)\cdots}$ (with the numerator of smaller degree than the denominator; and with the r_j distinct) can be written as a sum of terms of the form $\frac{A_j}{x - r_j}$ for suitable constants A_j.

In our case, this tells us that $\frac{1}{P(1 - P/M)} = \frac{A}{P} + \frac{B}{1 - P/M}$ for certain constants A and B.

Multiply both sides by P and set $P = 0$ to find $A = 1$.

Multiply both sides by $1 - P/M$ and set $P = M$ to find $B = 1/M$. This is what we used above.

The **logistic equation** with growth rate k and carrying capacity M is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right).$$

The general solution is $P(t) = \frac{M}{1 + Ce^{-kt}}$ where $C = \frac{M}{P(0)} - 1$.

Armin Straub
straub@southalabama.edu

Example 46. Let $P(t)$ describe the size of a population at time t. Under the logistic model of population growth, what is $\lim_{t\to\infty} P(t)$?

Solution.

- If $k>0$, then $e^{-kt}\to 0$ and it follows from $P(t)=\frac{M}{1+Ce^{-kt}}$ that $\lim_{t\to\infty} P(t)=M$.
  In other words, the population will approach the carrying capacity in the long run.
- If $k=0$, then we simply have $P(t)=\frac{M}{1+C}$. In other words, the population remains constant.
  This is a corner case because the DE becomes $\frac{\mathrm{d}P}{\mathrm{d}t}=0$.
- If $k<0$, then $e^{-kt}\to\infty$ and it follows that $\lim_{t\to\infty} P(t)=0$.
  In other words, the population will approach extinction in the long run.

Example 47. (homework) A rising population is modeled by the equation $\frac{\mathrm{d}P}{\mathrm{d}t}=400P-2P^2$.

(a) When the population size stabilizes in the long term, how big will the population be?

(b) Under which condition will the population size shrink?

(c) What is the population size when it is growing the fastest?

(d) If $P(0)=10$, what is $P(t)$?

Solution.

(a) Once the population reaches a stable level in the long term, we have $\frac{\mathrm{d}P}{\mathrm{d}t}=0$ (no change in population size).
Hence, $0=400P-2P^2=2P(200-P)$ which implies that $P=0$ or $P=200$. Since the population is rising, it will approach 200 in the long term.
Alternatively. Our DE matches the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t}=kP\left(1-\frac{P}{M}\right)$ with $k=400$ and $M=200$.

(b) The population size will shrink if $\frac{\mathrm{d}P}{\mathrm{d}t}<0$.
The DE tells us that is the case if and only if $400P-2P^2<0$ or, equivalently, if $P>\frac{400}{2}=200$.
Comment. In the logistic model, the population shrinks if it exceeds the carrying capacity.

(c) This is asking when $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the population growth) is maximal.
The DE is telling us that this growth is $f(P)=400P-2P^2$. This a parabola that opens to the bottom. From Calculus, we know that it has a global maximum when $f'(P)=0$.
$f'(P)=400-4P=0$ leads to $P=100$.
Thus, the population is growing the fastest when its size is 100.
Comment. In the logistic model, the population is growing fastest when it is half the carrying capacity.

(d) We know that the general solution of the logistic equation is $P(t)=\frac{M}{1+Ce^{-kt}}=\frac{200}{1+Ce^{-400t}}$.
Using $P(0)=10$, we find that $C=\frac{200}{10}-1=19$.
Thus $P(t)=\frac{200}{1+19e^{-400t}}$.

Example 48. A scientist is claiming that a certain population $P(t)$ follows the logistic model of population growth. How many data points do you need to begin to verify that claim?

Solution. The general solution $P(t)=\frac{M}{1+Ce^{-kt}}$ to the logistic equation has 3 parameters.

Hence, we need 3 data points just to solve for their values.

Once we have 4 or more data points, we are able to test whether $P(t)$ conforms to the logistic model.

Important comment. Complicated models tend to have many degrees of freedom, which makes it easier to fit them to real world data (even if the model is not actually particularly appropriate). We therefore need to be cognizant about how much evidence is needed to decide that a given model is appropriate for the data.

Further population models

Let $P(t)$ be the size of the population that we wish to model at time t.

Denote with $\beta(t)$ and $\delta(t)$ the birth and death rate at time t, measured in number of births or deaths per unit of population per unit of time.

In the time interval $[t, t + \Delta t]$, we have that

$$\Delta P \approx \beta(t)P(t)\Delta t - \delta(t)P(t)\Delta t.$$

Comment. The reason that this is not an exact equation is that the rates $\beta(t)$ and $\delta(t)$ are allowed to change with t. In the above, we used these rates at time t for all times in $[t, t + \Delta t]$. This is a good approximation if Δt is small.

Divide both sides by Δt and let $\Delta t \to 0$ to obtain the general differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta(t) - \delta(t))P.$$

Given certain scenarios, we now make corresponding reasonable choices for $\beta(t)$ and $\delta(t)$.

- **(basic)** If the rates $\beta(t)$ and $\delta(t)$ are constant over time, the DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta - \delta)P$.
  This is the exponential model of population growth.

- **(limited supply)** If supply is limited, the birth rate will decrease as P increases. The simplest such relationship would be a linear dependence, which would take the form $\beta(t) = \beta_0 - \beta_1 P$.
  On the other hand, we still assume that $\delta(t)$ is constant. (However, depending on circumstances, it could also be reasonable to assume that $\delta(t)$ increases as P increases.)
  With these assumptions, the corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta_0 - \beta_1 P - \delta)P$.
  This is the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(1 - P/M)$ with $k = \beta_0 - \delta$ and $\frac{k}{M} = \beta_1$.

- **(rare isolated species)** If the population consists of rare and isolated specimen which rely on chance encounters to reproduce, then it is reasonable to assume that the birth rate $\beta(t)$ is proportional to $P(t)$ (larger $P(t)$ means more possibilities for chance encounters). Once more, we assume that $\delta(t)$ constant.
  With these assumptions, the corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (kP - \delta)P$.
  This is, again, the logistic equation.

- **(rare isolated species with very long life)** As before, for a rare isolated population, it is reasonable to assume that $\beta(t)$ is proportional to $P(t)$. If, in addition, our specimen have very long life, then we would assume that $\delta(t) = 0$.
  The corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP^2$. Solutions are $P(t) = \frac{1}{C - kt}$ where $P(0) = 1/C$. (Do it!)
  Comment. Note that $P(t) \to \infty$ as $t \to C/k$. This explosion (which implies population growth beyond exponential growth) emphasizes that we can only use the DE while our initial assumptions are satisfied. Here, the DE is no longer appropriate when our species is no longer rare because $P(t)$ is too large.

- **(spread of contagious incurable virus)** Let $P(t)$ count the number of infected population units among a (constant) total of N. Since the virus is incurable, we have $\delta(t) = 0$. On the other hand, it is reasonable to assume that $\beta(t)$ is proportional to $N - P$ (the number of people that can still be infected).
  The resulting DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(N - P)$. Once again, this is the logistic equation.

- **(harvesting)** Suppose that h population units are harvested each unit of time.
  Then the DE becomes $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta(t) - \delta(t))P - h$.
  For instance. $\frac{\mathrm{d}P}{\mathrm{d}t} = kP - h$ has the solution $P(t) = Ce^{kt} + h/k$. In that case, we get exponential growth if $C > 0$. Note that $P(0) = C + h/k$. In terms of the initial population $P(0)$, we therefore get exponential growth if $P(0) > h/k$. (Also see next example!)

Example 49. A biotech company is growing certain microorganisms in the lab. From experience they know that the growth (number of organisms per day) of the microorganisms is well modeled by an exponential model with proportionality constant $k=5$ (per day). What is the optimal rate (in number of organisms per day) at which the company can harvest the microorganisms?

Solution. (long version via solving the DE) Without harvesting, the growth is modeled by $\frac{\mathrm{d}P}{\mathrm{d}t}=5P$ (the exponential model). Here, P is the number of organisms and t measures time in days. (Always think about your units in applications!)

If harvesting occurs at the rate of h number of organisms per day, the population model needs to be adjusted to

$$\frac{\mathrm{d}P}{\mathrm{d}t}=5P-h.$$

Since h is a constant, we can solve this DE using separation of variables. Alternatively, the DE is linear and we can therefore solve it using an integrating factor. For practice, we do both:

- **(separation of variables)** Integrating $\frac{1}{5P-h}\mathrm{d}P=\mathrm{d}t$, we find $\frac{1}{5}\ln|5P-h|=t+C$, which we simplify to $|5P-h|=e^{5t+5C}$. It follows that $5P-h=\pm e^{5t}e^{5C}=Be^{5t}$ where we wrote $B=\pm e^{5C}$ (note that the sign is fixed and cannot change).
  Thus, the general solution of the DE is $P(t)=\frac{h}{5}+Ae^{5t}$ (where we wrote $A=\frac{B}{5}$).
- **(integrating factor)** Since this is a linear DE, we can solve it as follows:
  - We write the DE in the form $\frac{\mathrm{d}P}{\mathrm{d}t}-5P=-h$.
  - The integrating factor is $f(t)=\exp(\int -5\,\mathrm{d}t)=e^{-5t}$.
  - Multiply the (rewritten) DE by $f(t)$ to get $\underbrace{e^{-5t}\frac{\mathrm{d}P}{\mathrm{d}t}-5e^{-5t}P}_{=\frac{\mathrm{d}}{\mathrm{d}t}[e^{-5t}P]}=-he^{-5t}$.
  - Integrate both sides to get $e^{-5t}P=-h\int e^{-5t}\mathrm{d}t=\frac{h}{5}e^{-5t}+C$.

  Hence the general solution to the DE is $P(t)=\frac{h}{5}+Ce^{5t}$.

In either case, we found that $P(t)=\frac{h}{5}+Ce^{5t}$. In order to be able to continually harvest, we need to make sure that $C\geqslant 0$. In terms of the initial population, we get $P(0)=\frac{h}{5}+C$ so that $C=P(0)-\frac{h}{5}$.

Thus the condition $C\geqslant 0$ becomes $P(0)-\frac{h}{5}\geqslant 0$ or, equivalently, $h\leqslant 5P(0)$. Thus, the optimal rate of harvesting is $h=5P(0)$.

Solution. (short version) As before, we observe that, if harvesting occurs at the rate of h number of organisms per day, then our population model is

$$\frac{\mathrm{d}P}{\mathrm{d}t}=5P-h.$$

In order to be able to continually harvest, we need to make sure that $\frac{\mathrm{d}P}{\mathrm{d}t}\geqslant 0$ (clearly, this is sufficient; we can also see that it is necessary since a decreased population should result in a lower optimal harvesting rate).

We thus get the condition $5P-h\geqslant 0$. Since the population is not decreasing (because $\frac{\mathrm{d}P}{\mathrm{d}t}\geqslant 0$), this is equivalent to $5P(0)-h\geqslant 0$ or, equivalently, $h\leqslant 5P(0)$. Again, we conclude that the optimal rate of harvesting is $h=5P(0)$.

Review. The **logistic equation** is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$.

Here, k is the growth rate and M is the carrying capacity.

The general solution is $P(t) = \frac{M}{1 + Ce^{-kt}}$ where $C = \frac{M}{P(0)} - 1$.

Example 50. In a city with a fixed population N, the time rate of change of the number P of people who have heard a certain rumor is proportional to the product of P and $N - P$. Suppose initially 10% have heard the rumor and after a week this number has grown to 20%. What percentage will this number reach after one more week?

Solution. $\frac{\mathrm{d}P}{\mathrm{d}t} = \gamma P(N - P)$. $P(0) = 0.1N$ and $P(1) = 0.2N$. We need $P(2)$.

Note that this is a logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$ with $k = \gamma N$ and carrying capacity N.

It therefore has the general solution $P(t) = \frac{N}{1 + Ce^{-kt}}$.

Using $P(0) = \frac{N}{1 + C} = 0.1N$, we find that $C = 9$.

Using $P(1) = \frac{N}{1 + 9e^{-k}} = 0.2N$, we further find that $e^{-k} = \frac{4}{9}$.

We could solve for k but note that it is more pleasing to use $e^{-kt} = (e^{-k})^t = \left(\frac{4}{9}\right)^t$ in our formula for $P(t)$.

We conclude that $P(t) = \frac{N}{1 + 9\left(\frac{4}{9}\right)^t}$.

In particular, $P(2) = \frac{N}{1 + 9 \cdot \frac{16}{81}} = \frac{9}{25}N$ which is 36%.

Armin Straub
straub@southalabama.edu

Mixing problems

Example 51. A tank contains 20gal of pure water. It is filled with brine (containing 5lb/gal salt) at a rate of 3gal/min. At the same time, well-mixed solution flows out at a rate of 2gal/min. How much salt is in the tank after t minutes?

Solution.

(Part I. determining a DE) Let $x(t)$ denote the amount of salt (in lb) in the tank after time t (in min).

At time t, the concentration of salt (in lb/gal) in the tank is $\frac{x(t)}{V(t)}$ where $V(t) = 20 + (3-2)t = 20 + t$ is the volume (in gal) in the tank.

In the time interval $[t, t+\Delta t]$: $\Delta x \approx 3 \cdot 5 \cdot \Delta t - 2 \cdot \frac{x(t)}{V(t)} \cdot \Delta t$.

Hence, $x(t)$ solves the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = 15 - 2 \cdot \frac{x}{20+t}$ with $x(0) = 0$.

Comment. Can you explain why the equation for Δx is only approximate but why the final DE is exact? [Hint: $x(t)/V(t)$ is the concentration at time t but we are using it for Δx at other times as well.]

(Part II. solving the DE) Since this is a linear DE, we can solve it as follows:

- Write the DE in the standard form as $\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{2}{20+t}x = 15$.
- The integrating factor is $f(t) = \exp\left(\int \frac{2}{20+t}\mathrm{d}t\right) = \exp(2\ln|20+t|) = (20+t)^2$.
- Multiply the DE (in standard form) by $f(t) = (20+t)^2$ to get $\underbrace{(20+t)^2\frac{\mathrm{d}x}{\mathrm{d}t} + 2(20+t)x}_{=\frac{\mathrm{d}}{\mathrm{d}t}[(20+t)^2 x]} = 15(20+t)^2$.
- Integrate both sides to get $(20+t)^2 x = 15\int (20+t)^2 \mathrm{d}t = 5(20+t)^3 + C$.

Hence the general solution to the DE is $x(t) = 5(20+t) + \frac{C}{(20+t)^2}$. Using $x(0) = 0$, we find $C = -5 \cdot 20^3$.

We conclude that, after t minutes, the tank contains $x(t) = 5(20+t) - \frac{5 \cdot 20^3}{(20+t)^2}$ pounds of salt.

Comment. As a consequence, $x(t) \approx 5(20+t) = 5V(t)$ for large t. Can you explain why that makes perfect sense and why we could have predicted this from the very beginning (without deriving a DE and solving it)?

Armin Straub
straub@southalabama.edu

Acceleration–velocity models

To model a falling object, we let $y(t)$ be its height at time t.

Then physics has names for $y'(t)$ and $y''(t)$: these are the **velocity** and the **acceleration**.

Physics tells us that objects fall due to gravity (and that it makes already-falling objects fall faster; in other words, gravity accelerates falling objects). Physicists have measured that, on earth, the the gravitational acceleration is $g \approx 9.81 \mathrm{m/s}^2$.

If we only take earth's gravitation into account, then the fall is therefore modeled by

$$y''(t) = -g.$$

Example 52. A ball is dropped from a 100m tall building. How long until it reaches the ground? What is the speed when it hits the ground?

Solution. Let $y(t)$ be the height (in meters) at which the ball is at time t (in seconds).

As above, physics tells us that an object falling due to gravity (and ignoring everything else) satisfies the DE $y'' = -g$ where $g \approx 9.81$. We further know the initial values $y(0) = 100$, $y'(0) = 0$.

Substituting $v = y'$ in the DE, we get $v' = -g$. This DE is solved by $v(t) = -gt + C$.

Hence, $y(t) = \int v(t) \mathrm{d}t = -\frac{1}{2} g t^2 + Ct + D$.

The initial conditions $y(0) = 100$, $y'(0) = 0$ tell us that $D = 100$ and $C = 0$.

Thus $y(t) = -\frac{1}{2} g t^2 + 100$.

The ball reaches the ground when $y(t) = -\frac{1}{2} g t^2 + 100 = 0$, that is after $t = \sqrt{200/g} \approx 4.52$ seconds.

The speed then is $|y'(4.52)| \approx 44.3 \,\mathrm{m}/s$.

For many applications, one needs to take air resistance into account.

This is actually less well understood than one might think, and the physics quickly becomes rather complicated. Typically, air resistance is somewhere in between the following two cases:

- Under certain assumptions, physics suggests that air resistance is proportional to the square of the velocity.

  Comment. A simplistic way to think about this is to imagine the falling object to bump into (air) particles; if the object falls twice as fast, then the momentum of the particles it bumps into is twice as large and it bumps into twice as many of them.

- In other cases such as "relatively slowly" falling objects, one might empirically observe that air resistance is proportional to the velocity itself.

  Comment. One might imagine that, at slow speed, the falling object doesn't exactly bump into particles but instead just gently pushes them aside; so that at twice the speed it only needs to gently push twice as often.

Example 53. When modeling the (slow) fall of a parachute, physics suggests that the air resistance is roughly proportional to velocity. If $y(t)$ is the parachute's height at time t, then the corresponding DE is $y'' = -g - \rho y'$ where $\rho > 0$ is a constant.

Comment. Note that $-\rho y' > 0$ because $y' < 0$. Thus, as intended, air resistance is acting in the opposite direction as gravity and slowing down the fall.

Determine the general solution of the DE.

Solution. Substituting $v = y'$, the DE becomes $v' + \rho v = -g$.

This is a linear DE. To solve it, we determine that the integrating factor is $\exp(\int \rho \mathrm{d}t) = e^{\rho t}$.

Multiplying the DE with that factor and integrating, we obtain $e^{\rho t} v = \int -g e^{\rho t} \mathrm{d}t = -\frac{g}{\rho} e^{\rho t} + C$.

Hence, $v(t) = -\frac{g}{\rho} + C e^{-\rho t}$.

Correspondingly, the general solution of the DE is $y(t) = \int v(t) \mathrm{d}t = -\frac{g}{\rho} t - \frac{C}{\rho} e^{-\rho t} + D$.

Comment. Note that $\lim_{t \to \infty} v(t) = -\frac{g}{\rho}$. In other words, the **terminal velocity** is $v_\infty = -\frac{g}{\rho}$.

This is an interesting mathematical consequence of the DE. (And important for the idea behind a parachute!)

Note that, if we know that there is a terminal speed, then we can actually determine its value v_∞ from the DE without solving it by setting $v' = 0$ (because, once the terminal speed is reached, the velocity does not change anymore) in $v' + \rho v = -g$. This gives us $\rho v_\infty = -g$ and, hence, $v_\infty = -g / \rho$ as above.

Numerically "solving" DEs: Euler's method

Recall that the general form of a first-order initial value problem is

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Further recall that, under mild assumptions on $f(x, y)$, such an IVP has a unique solution $\psi(x)$.

Comment. While deriving Euler's method, we write $\psi(x)$ instead of $y(x)$ simply to not confuse ourselves (note that y is also used as the variable in the differential equation and y will also be used as the variable when writing down equations for tangent lines).

We have learned some techniques for (exactly) solving DEs. On the other hand, many DEs that arise in practice cannot be solved by these techniques (or more fancy ones).

Instead, it is common in practice to approximate the solution $\psi(x)$ to our IVP. Euler's method is the simplest example of how this can done. The key idea is to locally approximate $\psi(x)$ by tangent lines.

Example 54. Spell out the equation for the tangent line of a function $\psi(x)$ at the point (x_0, y_0).

Solution. $y = y_0 + \psi'(x_0)(x - x_0)$

Note that, in our case where $\psi(x)$ solves the IVP, we know what $\psi'(x_0)$ is! Namely,

$$\psi'(x_0) = f(x_0, \psi(x_0)) = f(x_0, y_0).$$

Comment. Make sure you see that the right-hand is a number that we can easily compute.

We therefore know the equation of the tangent line of $\psi(x)$ at the initial point (x_0, y_0). It is

$$y = y_0 + f(x_0, y_0)(x - x_0).$$

The idea is to use this tangent line as an approximation of $\psi(x)$ for just a little bit, namely from x_0 to $x_0 + h$ for a small **step size** h. We set $x_1 = x_0 + h$ and compute y_1 as the corresponding y-value on the tangent line using

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0) = \boxed{y_0 + h f(x_0, y_0)}.$$

Note that $y_1 \approx \psi(x_1)$ is a good approximation if h is sufficiently small.

At this point, we have gone from our initial point (x_0, y_0) to a next (approximate) point (x_1, y_1). We now repeat what we did to get a third point (x_2, y_2) with $x_2 = x_1 + h$. Continuing in this way, we obtain Euler's method:

(Euler's method) To approximate the solution $y(x)$ of the IVP $y' = f(x, y)$, $y(x_0) = y_0$, we start with the point (x_0, y_0) and a step size h. We then compute:

$$\begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + h f(x_n, y_n) \end{aligned}$$

Example 55. Consider the IVP $\frac{\mathrm{d}y}{\mathrm{d}x} = (2x-3y)^2 + \frac{2}{3}$, $y(1) = \frac{1}{3}$.

(a) Approximate the solution $y(x)$ for $x \in [1,2]$ using Euler's method with 2 steps.

(b) Approximate the solution $y(x)$ for $x \in [1,2]$ using Euler's method with 3 steps.

(c) Solve this IVP exactly. Compare the values at $x=2$.

Solution.

(a) The step size is $h = \frac{2-1}{2} = \frac{1}{2}$. We apply Euler's method with $f(x,y) = (2x-3y)^2 + \frac{2}{3}$:

$$\begin{array}{ll} x_0 = 1 & y_0 = \frac{1}{3} \\ x_1 = \frac{3}{2} & y_1 = y_0 + h f(x_0, y_0) = \frac{1}{3} + \frac{1}{2} \cdot \left[\left(2 \cdot 1 - 3 \cdot \frac{1}{3}\right)^2 + \frac{2}{3}\right] = \frac{7}{6} \\ x_2 = 2 & y_2 = y_1 + h f(x_1, y_1) = \frac{7}{6} + \frac{1}{2} \cdot \frac{11}{12} = \frac{13}{8} \end{array}$$

In particular, the approximation for $y(2)$ is $y_2 = \frac{13}{8} = 1.625$.

(b) The step size is $h = \frac{2-1}{3} = \frac{1}{3}$. We again apply Euler's method with $f(x,y) = (2x-3y)^2 + \frac{2}{3}$:

$$\begin{array}{ll} x_0 = 1 & y_0 = \frac{1}{3} \\ x_1 = \frac{4}{3} & y_1 = y_0 + h f(x_0, y_0) = \frac{1}{3} + \frac{1}{3} \cdot \left[\left(2 \cdot 1 - 3 \cdot \frac{1}{3}\right)^2 + \frac{2}{3}\right] = \frac{8}{9} \\ x_2 = \frac{5}{3} & y_2 = y_1 + h f(x_1, y_1) = \frac{8}{9} + \frac{1}{3} \cdot \frac{2}{3} = \frac{10}{9} \\ x_3 = 2 & y_3 = y_2 + h f(x_2, y_2) = \frac{10}{9} + \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{3} \end{array}$$

In particular, the approximation for $y(2)$ is $y_3 = \frac{4}{3} \approx 1.333$.

(c) We solved this IVP in Example 37 using the substitution $u = 2x - 3y$ followed by separation of variables. We found that the unique solution of the IVP is $y(x) = \frac{2}{3}x - \frac{1}{3(3x-2)}$.

In particular, the exact value at $x=2$ is $y(2) = \frac{5}{4} = 1.25$.

We observe that our approximations for $y(2) = 1.25$ improved from 1.625 to 1.333 as we increased the number of steps (equivalently, we decreased the step size h from $\frac{1}{2}$ to $\frac{1}{3}$).

For comparison. With 10 steps (so that $h = \frac{1}{10}$), the approximation improves to $y(2) \approx 1.259$.

Example 56. Consider the IVP $y' = y$, $y(0) = 1$. Approximate the solution $y(x)$ for $x \in [0,1]$ using Euler's method with 4 steps. In particular, what is the approximation for $y(1)$?

Comment. Of course, the real solution is $y(x) = e^x$. In particular, $y(1) = e \approx 2.71828$.

Solution. The step size is $h = \frac{1-0}{4} = \frac{1}{4}$. We apply Euler's method with $f(x,y) = y$:

$$\begin{array}{ll} x_0 = 0 & y_0 = 1 \\ x_1 = \frac{1}{4} & y_1 = y_0 + h f(x_0, y_0) = 1 + \frac{1}{4} \cdot 1 = \frac{5}{4} = 1.25 \\ x_2 = \frac{1}{2} & y_2 = y_1 + h f(x_1, y_1) = \frac{5}{4} + \frac{1}{4} \cdot \frac{5}{4} = \frac{5^2}{4^2} = 1.5625 \\ x_3 = \frac{3}{4} & y_3 = y_2 + h f(x_2, y_2) = \frac{5^2}{4^2} + \frac{1}{4} \cdot \frac{5^2}{4^2} = \frac{5^3}{4^3} \approx 1.9531 \\ x_4 = 1 & y_4 = y_3 + h f(x_3, y_3) = \frac{5^3}{4^3} + \frac{1}{4} \cdot \frac{5^3}{4^3} = \frac{5^4}{4^4} \approx 2.4414 \end{array}$$

In particular, the approximation for $y(1)$ is $y_4 \approx 2.4414$.

Comment. Can you see that, if instead we start with $h = \frac{1}{n}$, then we similarly get $x_i = \frac{(n+1)^i}{n^i}$ for $i = 0, 1, \ldots, n$? In particular, $y(1) \approx y_n = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n \to e$ as $n \to \infty$. Do you recall how to derive this final limit?

Preview: Solving linear differential equations with constant coefficients

Let us have another look at Example 11. Note that the DE is a second-order linear differential equation with constant coefficients. Our upcoming goal will be to solve all such equations.

Example 57. Find the general solution to $y'' = y' + 6y$.

Solution. We look for solutions of the form e^{rx}.

Plugging e^{rx} into the DE, we get $r^2 e^{rx} = r e^{rx} + 6 e^{rx}$ which simplifies to $r^2 - r - 6 = 0$.

This is called the **characteristic equation**. Its solutions are $r = -2, 3$ (the **characteristic roots**).

This means we found the two solutions $y_1 = e^{-2x}$, $y_2 = e^{3x}$.

The general solution to the DE is $C_1 e^{-2x} + C_2 e^{3x}$.

Comment. In the final step, we used an important principle that is true for linear (!) homogeneous DEs. Namely, if we have solutions $y_1, y_2, \ldots$ then any linear combination $C_1 y_1 + C_2 y_2 + \ldots$ is a solution as well. We will discuss this soon but, for now, check that $C_1 e^{-2x} + C_2 e^{3x}$ is indeed a solution by plugging it into the DE.

Example 58. (extra) Find the general solution to $y''' = y'' + 6y'$.

Solution. We look for solutions of the form e^{rx}.

Plugging e^{rx} into the DE, we get $r^3 e^{rx} = r^2 e^{rx} + 6r e^{rx}$ which simplifies to $r^3 - r^2 - 6r = r(r^2 - r - 6) = 0$.

As in Example 57, $r^2 - r - 6 = 0$ has the solutions $r = -2, 3$.

Overall, $r(r^2 - r - 6) = 0$ has the three solutions $-2, 3, 0$.

This means we found the three solutions $y_1 = e^{-2x}$, $y_2 = e^{3x}$, $y_3 = e^{0x} = 1$.

The general solution to the DE is $C_1 e^{-2x} + C_2 e^{3x} + C_3$.

Alternatively. We can substitute $u = y'$, in which case the new DE is $u'' = u' + 6u$. From Example 57, we know that $u = C_1 e^{-2x} + C_2 e^{3x}$.

Hence, the general solution of the initial DE is $y = \int u \, dx = -\frac{1}{2} C_1 e^{-2x} + \frac{1}{3} C_2 e^{3x} + C$.

Note that we can set $D_1 = -\frac{1}{2} C_1$, $D_2 = \frac{1}{3} C_2$, $D_3 = C$ to write this as $D_1 e^{-2x} + D_2 e^{3x} + D_3$, which matches our earlier solution.

Spotlight on the exponential function

Example 59. Solve $y' = ky$ where k is a constant.

Solution. (experience) At this point, we can probably see that $y(x) = e^{kx}$ is a solution.
In fact, the general solution is $y(x) = Ce^{kx}$.
That there cannot be any further solutions follows from the existence and uniqueness theorem (see next example).

Solution. (separation of variables) Alternatively, we can solve the DE using separation of variables.

Express the DE as $\frac{dy}{dx} = ky$, then write it as $\frac{1}{y}dy = k\,dx$ (note that we just lost the solution $y = 0$).

Integrating gives $\ln|y| = kx + D$, hence $|y| = e^{kx+D}$.

Since the RHS is never zero, $y = \pm e^{kx+D} = Ce^{kx}$ (with $C = \pm e^D$). Finally, note that $C = 0$ corresponds to the singular solution $y = 0$ that we lost. In summary, the general solution is Ce^{kx}.

Example 60. Consider the IVP $y' = ky$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = ky$. We compute that $\frac{\partial}{\partial y} f(x, y) = k$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y).

Hence, for any initial conditions, the IVP locally has a unique solution by the existence and uniqueness theorem.

Comment. As a consequence, there can be no other solutions to the DE $y' = ky$ than the ones of the form $y(x) = Ce^{kx}$. Why?! [Assume that $y(x)$ satisfies $y' = ky$ and let (a, b) any value on the graph of y. Then $y(x)$ solves the IVP $y' = ky$, $y(a) = b$; but so does Ce^{kx} with $C = b / e^{ka}$. The uniqueness implies that $y(x) = Ce^{kx}$.]

In particular, we have the following characterization of the exponential function:

e^x is the unique solution to the IVP $y' = y$, $y(0) = 1$.

Comment. Note that, for instance, $\frac{d}{dx} 2^x = \ln(2)\, 2^x$. (This follows from $2^x = e^{\ln(2^x)} = e^{x \ln(2)}$.)
Since $\ln = \log_e$, this means that we cannot avoid the natural base $e \approx 2.718$ even if we try to use another base.

Euler's method applied to e^x

Example 61. (cont'd) Consider the IVP $y' = y$, $y(0) = 1$. Approximate the solution $y(x)$ for $x \in [0, 1]$ using Euler's method with n steps for several values of n. In each case, what is the approximation for $y(1)$?

Solution. Since the real solution is $y(x) = e^x$ so that, in particular, the exact solution is $y(1) = e \approx 2.71828$.
We proceed as we did in Example 56 in the case $n = 4$ and apply Euler's method with $f(x, y) = y$:

$$\begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + h \underbrace{f(x_n, y_n)}_{= y_n} = (1 + h) y_n \end{aligned}$$

We observe that it follows from $y_{n+1} = (1 + h) y_n$ that $y_n = (1 + h)^n y_0$. Since $y_0 = 1$ and $h = \frac{1 - 0}{n} = \frac{1}{n}$, we conclude that

$$x_n = 1, \quad y_n = \left(1 + \frac{1}{n}\right)^n.$$

[For instance, for $n = 4$, we get $x_4 = 1$, $y_4 = \left(\frac{5}{4}\right)^4 \approx 2.4414$ as in Example 56.]

In particular, our approximation for $y(1)$ is $\left(1 + \frac{1}{n}\right)^n$.

Armin Straub
straub@southalabama.edu

Here are a few values spelled out:

$$\begin{aligned}
n=1\colon &\quad \left(1+\frac{1}{n}\right)^n = 2\\
n=4\colon &\quad \left(1+\frac{1}{n}\right)^n = 2.4414\ldots\\
n=12\colon &\quad \left(1+\frac{1}{n}\right)^n = 2.6130\ldots\\
n=100\colon &\quad \left(1+\frac{1}{n}\right)^n = 2.7048\ldots\\
n=365\colon &\quad \left(1+\frac{1}{n}\right)^n = 2.7145\ldots\\
n=1000\colon &\quad \left(1+\frac{1}{n}\right)^n = 2.7169\ldots\\
n\to\infty\colon &\quad \left(1+\frac{1}{n}\right)^n \to e = 2.71828\ldots
\end{aligned}$$

We can see that Euler's method converges to the correct value as $n\to\infty$. On the other hand, we can see that it doesn't converge impressively fast. That is why, for serious applications, one usually doesn't use Euler's method directly but rather higher-order methods derived from the same principles (such as Runge–Kutta methods).

Interpretation. Note that we can interpret the above values in terms of compound interest. We start with initial capital of $y(0)=1$ and we are interested in the capital $y(1)$ after 1 year if we receive interest at an annual rate of 100%:

- If we receive a single interest payment at the end of the year, then $y(1)=2$ (case $n=1$ above).
- If we receive quarterly interest payments of $\frac{100\%}{4}=25\%$ each, then $y(1)=(1.25)^4=2.441\ldots$ (case $n=4$).
- If we receive monthly interest payments of $\frac{100\%}{12}=\frac{1}{12}$ each, then $y(1)=2.6130\ldots$ (case $n=12$).
- If we receive daily interest payments of $\frac{100\%}{365}=\frac{1}{365}$ each, then $y(1)=2.7145\ldots$ (case $n=365$).

It is natural to wonder what happens if interest payments are made more and more frequently. Well, we already know the answer! If interest is compounded continuously, then we have e in our bank account after one year.

Challenge. Can you evaluate the limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ using your Calculus I skills?

Linear DEs of higher order

The most general linear first-order DE is of the form $A(x)y' + B(x)y + C(x) = 0$. Any such DE can be rewritten in the form $y' + P(x)y = f(x)$ by dividing by $A(x)$ and rearranging.

We have learned how to solve all of these using an integrating factor.

Likewise, any **linear DE** of order n can be written in the form

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = f(x).$$

The corresponding **homogeneous linear DE** is the DE

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = 0,$$

and it plays an important role in solving the original linear DE.

A linear DE is **homogeneous** if and only if the zero function $y(x) = 0$ is a solution.

Advanced comment. As we observed in the first-order case, if I is an interval on which all the $P_j(x)$ as well as $f(x)$ are continuous, then for any $a \in I$ the IVP with $y(a) = b_0$, $y'(a) = b_1$, ..., $y^{(n-1)}(a) = b_{n-1}$ always has a unique solution (which is defined on all of I).

(general solution of linear DEs) For a linear DE of order n, the general solution always takes the form

$$y(x) = y_p(x) + C_1 y_1(x) + \ldots + C_n\, y_n(x),$$

where y_p is any solution (called a **particular solution**) and y_1, y_2, ..., y_n are solutions to the corresponding **homogeneous** linear DE.

Comment. If the linear DE is already homogeneous, then the zero function $y(x) = 0$ is a solution and we can use $y_p = 0$. In that case, the general solution is of the form $y(x) = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$.

Why? This structure of the solution follows from the observation in the next example.

Example 62. Suppose that y_1 solves $y'' + P(x)y' + Q(x)y = f(x)$ and that y_2 solves $y'' + P(x)y' + Q(x)y = g(x)$ (note that the corresponding homogeneous DE is the same). Show that $7y_1 + 4y_2$ solves $y'' + P(x)y' + Q(x)y = 7f(x) + 4g(x)$.

Solution. $(7y_1 + 4y_2)'' + P(x)(7y_1 + 4y_2)' + Q(x)(7y_1 + 4y_2)$
$= 7\{y_1'' + P(x)y_1' + Q(x)y_1\} + 4\{y_2'' + P(x)y_2' + Q(x)y_2\} = 7 \cdot f(x) + 4 \cdot g(x)$

Comment. Of course, there is nothing special about the coefficients 7 and 4.

Important comment. In particular, if both $f(x)$ and $g(x)$ are zero, then $7f(x) + 4g(x)$ is zero as well. This shows that homogeneous linear DEs have the important property that, if y_1 and y_2 are two solutions, then any linear combination $C_1\, y_1 + C_2\, y_2$ is a solution as well.

The upshot is that this observation reduces the task of finding the general solution of a homogeneous linear DE to the task of finding n (sufficiently) different solutions.

Armin Straub
straub@southalabama.edu

Example 63. (extra) The DE $x^2y''+2xy'-6y=0$ has solutions $y_1=x^2$, $y_2=x^{-3}$.

(a) Determine the general solution

(b) Solve the IVP with $y(2)=10$, $y'(2)=15$.

Solution.

(a) Note that this is a homogeneous linear DE of order 2.
Hence, given the two solutions, we conclude that the general solution is $y(x)=Ax^2+Bx^{-3}$ (in this case, the particular solution is $y_p=0$ because the DE is homogeneous).

(b) Using $y'(x)=2Ax-3Bx^{-4}$, the two initial conditions allow us to solve for A and B:
Solving $y(2)=4A+B/8=10$ and $y'(2)=4A-3/16B=15$ for A and B results in $A=3$, $B=-16$.
So the unique solution to the IVP is $y(x)=3x^2-16/x^3$.

Homogeneous linear DEs with constant coefficients

Let us start with another example like Examples 11 and 57. This time we also approach this computation using an operator approach that explains further what is going on (and that will be particularly useful when we discuss inhomogeneous equations).

An **operator** takes a function as input and returns a function as output. That is exactly what the derivative does.

In the sequel, we write $D=\frac{\mathrm{d}}{\mathrm{d}x}$ for the derivative operator.

For instance. We write $y'=\frac{\mathrm{d}}{\mathrm{d}x}y=Dy$ as well as $y''=\frac{\mathrm{d}^2}{\mathrm{d}x^2}y=D^2y$.

Example 64. Find the general solution to $y''-y'-2y=0$.

Solution. (our earlier approach) Let us look for solutions of the form e^{rx}.
Plugging e^{rx} into the DE, we get $r^2e^{rx}-re^{rx}-2e^{rx}=0$.
Equivalently, $r^2-r-2=0$. This is the characteristic equation. Its solutions are $r=2,-1$.
This means we found the two solutions $y_1=e^{2x}$, $y_2=e^{-x}$.
Since this a homogeneous linear DE, the general solution is $y=C_1e^{2x}+C_2e^{-x}$.

Solution. (operator approach) $y''-y'-2y=0$ is equivalent to $(D^2-D-2)y=0$.
Note that $D^2-D-2=(D-2)(D+1)$ is the **characteristic polynomial**.
Observe that we get solutions to $(D-2)(D+1)y=0$ from $(D-2)y=0$ and $(D+1)y=0$.
$(D-2)y=0$ is solved by $y_1=e^{2x}$, and $(D+1)y=0$ is solved by $y_2=e^{-x}$; as in the previous solution.
Again, we conclude that the general solution is $y=C_1e^{2x}+C_2e^{-x}$.

Set $D=\frac{\mathrm{d}}{\mathrm{d}x}$. Every **homogeneous linear DE with constant coefficients** can be written as $p(D)y=0$, where $p(D)$ is a polynomial in D, called the **characteristic polynomial**.

For instance. $y''-y'-2y=0$ is equivalent to $Ly=0$ with $L=D^2-D-2$.

Example 65. Solve $y''-y'-2y=0$ with initial conditions $y(0)=4$, $y'(0)=5$.

Solution. From Example 64, we know that the general solution is $y(x)=C_1e^{2x}+C_2e^{-x}$.
Using $y'(x)=2C_1e^{2x}-C_2e^{-x}$, the initial conditions result in the two equations $C_1+C_2=4$, $2C_1-C_2=5$.
Solving these we find $C_1=3$, $C_2=1$.
Hence the unique solution to the IVP is $y(x)=3e^{2x}+e^{-x}$.

Example 66.

(a) Check that $y = -3x$ is a solution to $y'' - y' - 2y = 6x + 3$.

Comment. We will soon learn how to find such a solution from scratch.

(b) Using the first part, determine the general solution to $y'' - y' - 2y = 6x + 3$.

(c) Determine $f(x)$ so that $y = 7x^2$ solves $y'' - y' - 2y = f(x)$.

Comment. This is how you can create problems like the ones in the first two parts.

Solution.

(a) If $y = -3x$, then $y' = -3$ and $y'' = 0$. Plugging into the DE, we find $0 - (-3) - 2 \cdot (-3x) = 6x + 3$, which verifies that this is a solution.

(b) This is an inhomogeneous linear DE. From Example 64, we know that the corresponding homogeneous DE has the general solution $C_1 e^{2x} + C_2 e^{-x}$.
From the first part, we know that $-3x$ is a particular solution.
Combining this, the general solution to the present DE is $-3x + C_1 e^{2x} + C_2 e^{-x}$.

(c) If $y = 7x^2$, then $y' = 14x$ and $y'' = 14$ so that $y'' - y' - 2y = 14 - 14x - 14x^2$.
Thus $f(x) = 14 - 14x - 14x^2$.

Review. Every homogeneous linear DE with constant coefficients can be written as $p(D)y = 0$, where $D = \frac{\mathrm{d}}{\mathrm{d}x}$ and $p(D)$ is the characteristic polynomial polynomial. Each root r of the characteristic polynomial gives us one solution, namely $y = e^{rx}$, of the DE.

Example 67.

(a) Determine the general solution of $y''' + 7y'' + 14y' + 8y = 0$.

(b) Determine the general solution of $y^{(4)} + 7y''' + 14y'' + 8y' = 0$.

Solution.

(a) This DE is of the form $p(D)\,y = 0$ with characteristic polynomial $p(D) = D^3 + 7D^2 + 14D + 8$.
The characteristic polynomial factors as $p(D) = (D+1)(D+2)(D+4)$.
Hence, we found the solutions $y_1 = e^{-x}$, $y_2 = e^{-2x}$, $y_3 = e^{-4x}$. Those are enough (independent!) solutions for a third-order DE. The general solution therefore is $y(x) = C_1\,e^{-x} + C_2\,e^{-2x} + C_3\,e^{-4x}$.

(b) The DE now is of the form $p(D)\,y = 0$ with characteristic polynomial $p(D) = D(D^3 + 7D^2 + 14D + 8)$.
Hence, the characteristic polynomial factors as $p(D) = D(D+1)(D+2)(D+4)$ and we find the additional solution $y_4 = e^{0x} = 1$. Thus, the general solution is $y(x) = C_1\,e^{-x} + C_2\,e^{-2x} + C_3\,e^{-4x}$.
Comment. If we didn't know about roots of characteristic polynomials, an alternative approach would be to substitute $u = y'$, resulting in the DE $u''' + 7u'' + 14u' + 8u = 0$. From the first part, we know that $u(x) = C_1\,e^{-x} + C_2\,e^{-2x} + C_3\,e^{-4x}$. Hence, $y(x) = \int u(x)\mathrm{d}x = -C_1\,e^{-x} - \frac{1}{2}C_2\,e^{-2x} - \frac{1}{4}C_3\,e^{-4x} + C$.
Make sure you see that this is an equivalent way of presenting the general solution! (For instance, since C_3 can be any constant, it doesn't make a difference whether we write $-\frac{1}{4}C_3$ or C_3. The latter is preferable unless the $-\frac{1}{4}$ is useful for some purpose.)

Example 68. Determine the general solution of $y''' - y'' - 4y' + 4y = 0$.

Solution. This DE is of the form $p(D)\,y = 0$ with characteristic polynomial $p(D) = D^3 - D^2 - 4D + 4$.
The characteristic polynomial factors as $p(D) = (D-1)(D-2)(D+2)$.
Hence, we found the solutions $y_1 = e^x$, $y_2 = e^{2x}$, $y_3 = e^{-2x}$. Those are enough (independent!) solutions for a third-order DE. The general solution therefore is $y(x) = C_1\,e^x + C_2\,e^{2x} + C_3\,e^{-2x}$.

In this manner, we are able to solve any homogeneous linear DE of order n with constant coefficients provided that there are n different roots r (each giving rise to one solution e^{rx}).

One issue is that roots might be repeated. In that case, we are currently missing solutions. The following example suggests how to get our hands on the missing solutions.

Example 69. Determine the general solution of $y''' = 0$.

Solution. We know from Calculus that the general solution is $y(x) = C_1 + C_2\,x + C_3\,x^2$.

Solution. (looking ahead) The characteristic polynomial $p(D) = D^3$ has roots $0, 0, 0$. By Theorem 70 below, we have the solutions $y(x) = x^j\,e^{0x} = x^j$ for $j = 0, 1, 2$, so that the general solution is $y(x) = C_1 + C_2\,x + C_3\,x^2$.

Armin Straub
straub@southalabama.edu

Theorem 70. Consider the homogeneous linear DE with constant coefficients $p(D)y = 0$.

- If r is a root of the characteristic polynomial and if k is its multiplicity, then k (independent) solutions of the DE are given by $x^j e^{rx}$ for $j = 0, 1, \ldots, k-1$.
- Combining these solutions for all roots, gives the general solution.

This is because the order of the DE equals the degree of $p(D)$, and a polynomial of degree n has (counting with multiplicity) exactly n (possibly **complex**) roots.

In the complex case. Likewise, if $r = a \pm bi$ are roots of the characteristic polynomial and if k is its multiplicity, then $2k$ (independent) solutions of the DE are given by $x^j e^{ax}\cos(bx)$ and $x^j e^{ax}\sin(bx)$ for $j = 0, 1, \ldots, k-1$. This case will be discussed next time.

Proof. Let r be a root of the characteristic polynomial of multiplicity k. Then $p(D) = q(D)\,(D-r)^k$.

We need to find k solutions to the simpler DE $(D-r)^k y = 0$.

It is natural to look for solutions of the form $y = c(x)e^{rx}$.

[This idea is called **variation of constants** since we know that this is a solution if $c(x)$ is a constant.]

Note that $(D-r)[c(x)e^{rx}] = (c'(x)e^{rx} + c(x)re^{rx}) - rc(x)e^{rx} = c'(x)e^{rx}$.

Repeating, we get $(D-r)^2[c(x)e^{rx}] = (D-r)[c'(x)e^{rx}] = c''(x)e^{rx}$ and, eventually, $(D-r)^k[c(x)e^{rx}] = c^{(k)}(x)e^{rx}$.

In particular, $(D-r)^k y = 0$ is solved by $y = c(x)e^{rx}$ if and only if $c^{(k)}(x) = 0$.

The DE $c^{(k)}(x) = 0$ is clearly solved by x^j for $j = 0, 1, \ldots, k-1$, and it follows that $x^j e^{rx}$ solves the original DE. □

Example 71. Determine the general solution of $y''' - 3y' + 2y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - 3D + 2 = (D-1)^2(D+2)$ has roots $1, 1, -2$.

By Theorem 70, the general solution is $y(x) = (C_1 + C_2 x)e^x + C_2 e^{-2x}$.

Example 72. (homework) Solve the IVP $y''' = 4y'' - 4y'$ with $y(0) = 4$, $y'(0) = 0$, $y''(0) = -4$.

Solution. The characteristic polynomial $p(D) = D^3 - 4D^2 + 4D = D(D-2)^2$ has roots $0, 2, 2$.

By Theorem 70, the general solution is $y(x) = C_1 + (C_2 + C_3 x)e^{2x}$.

Using $y'(x) = (2C_2 + C_3 + 2C_3 x)e^{2x}$ and $y''(x) = 4(C_2 + C_3 + C_3 x)e^{2x}$, the initial conditions result in the equations $C_1 + C_2 = 4$, $2C_2 + C_3 = 0$, $4C_2 + 4C_3 = -4$.

Solving these (start with the last two equations) we find $C_1 = 3$, $C_2 = 1$, $C_3 = -2$.

Hence the unique solution to the IVP is $y(x) = 3 + (1 - 2x)e^{2x}$.

Armin Straub
straub@southalabama.edu

Review. A homogeneous linear DE with constant coefficients is of the form $p(D)y = 0$, where $p(D)$ is the characteristic polynomial polynomial. For each characteristic root r of multiplicity k, we get the k solutions $x^j e^{rx}$ for $j = 0, 1, \ldots, k-1$.

Example 73. Determine the general solution of $y''' - 3y'' + 3y' - y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - 3D^2 + 3D - 1 = (D-1)^3$ has roots $1, 1, 1$.
By Theorem 70, the general solution is $y(x) = (C_1 + C_2 x + C_3 x^2) e^x$.

Example 74. Determine the general solution of $y''' - y'' - 5y' - 3y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - D^2 - 5D - 3 = (D-3)(D+1)^2$ has roots $3, -1, -1$.
Hence, the general solution is $y(x) = C_1 e^{3x} + (C_2 + C_3 x) e^{-x}$.

Example 75. (homework) Solve the IVP $y''' = 8y'' - 16y'$ with $y(0) = 1$, $y'(0) = 4$, $y''(0) = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - 8D^2 + 16D = D(D-4)^2$ has roots $0, 4, 4$.
By Theorem 70, the general solution is $y(x) = C_1 + (C_2 + C_3 x) e^{4x}$.
Using $y'(x) = (4C_2 + C_3 + 4C_3 x) e^{4x}$ and $y''(x) = 4(4C_2 + 2C_3 + 4C_3 x) e^{4x}$, the initial conditions result in the equations $C_1 + C_2 = 1$, $4C_2 + C_3 = 4$, $16C_2 + 8C_3 = 0$.
Solving these (start with the last two equations) we find $C_1 = -1$, $C_2 = 2$, $C_3 = -4$.
Hence the unique solution to the IVP is $y(x) = -1 + (2 - 4x) e^{4x}$.
Important comment. Check that $y(x)$ indeed solves the IVP.

Example 76. Determine the general solution of $y^{(6)} = 3y^{(5)} - 4y'''$.

Solution. This DE is of the form $p(D)\, y = 0$ with $p(D) = D^6 - 3D^5 + 4D^3 = D^3 (D-2)^2 (D+1)$.
The characteristic roots are $2, 2, 0, 0, 0, -1$.
By Theorem 70, the general solution is $y(x) = (C_1 + C_2 x) e^{2x} + C_3 + C_4 x + C_5 x^2 + C_6 e^{-x}$.

Example 77. Consider the function $y(x) = 3x e^{-2x} + 7e^x$. Determine a homogeneous linear DE with constant coefficients of which $y(x)$ is a solution.

Solution. In order for $y(x)$ to be a solution of $p(D) y = 0$, the characteristic roots must include $-2, -2, 1$.
The simplest choice for $p(D)$ thus is $p(D) = (D+2)^2 (D-1) = D^3 + 3D^2 - 4$.
Accordingly, $y(x)$ is a solution of $y''' + 3y'' - 4y = 0$.

Example 78. Consider the function $y(x) = 3x e^{-2x} + 7$. Determine a homogeneous linear DE with constant coefficients of which $y(x)$ is a solution.

Solution. In order for $y(x)$ to be a solution of $p(D) y = 0$, the characteristic roots must include $-2, -2, 0$.
The simplest choice for $p(D)$ thus is $p(D) = (D+2)^2 D = D^3 + 4D^2 + 4D$.
Accordingly, $y(x)$ is a solution of $y''' + 4y'' + 4y' = 0$.

Armin Straub
straub@southalabama.edu

Real form of complex solutions

Let's recall some basic facts about **complex numbers**:

- Every complex number can be written as $z = x + iy$ with real x, y.
- Here, the imaginary unit i is characterized by solving $x^2 = -1$.

  Important observation. The same equation is solved by $-i$. This means that, algebraically, we cannot distinguish between $+i$ and $-i$.

- The **conjugate** of $z = x + iy$ is $\bar{z} = x - iy$.

  Important comment. Since we cannot algebraically distinguish between $\pm i$, we also cannot distinguish between z and $\bar{z}$. That's the reason why, in problems involving only real numbers, if a complex number $z = x + iy$ shows up, then its **conjugate** $\bar{z} = x - iy$ has to show up in the same manner. With that in mind, have another look at the examples below.

- The **real part** of $z = x + iy$ is x and we write $\operatorname{Re}(z) = x$.

  Likewise the **imaginary part** is $\operatorname{Im}(z) = y$.

  Observe that $\operatorname{Re}(z) = \frac{1}{2}(z + \bar{z})$ as well as $\operatorname{Im}(z) = \frac{1}{2i}(z - \bar{z})$.

Theorem 79. (Euler's identity) $e^{ix} = \cos(x) + i\sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] □

Comment. It follows that $\cos(x) = \operatorname{Re}(e^{ix}) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \operatorname{Im}(e^{ix}) = \frac{1}{2i}(e^{ix} - e^{-ix})$.

Example 80. Determine the general solution of $y'' + y = 0$.

Solution. (complex numbers in general solution) The characteristic polynomial is $D^2 + 1$ which has no roots over the reals. Over the **complex numbers**, by definition, the roots are i and $-i$.
So the general solution is $y(x) = C_1 e^{ix} + C_2 e^{-ix}$.

Solution. (real general solution) On the other hand, we easily check that $y_1 = \cos(x)$ and $y_2 = \sin(x)$ are two solutions. Hence, the general solution can also be written as $y(x) = D_1\cos(x) + D_2\sin(x)$.

Important comment. That we have these two different representations is a consequence of Euler's identity (Theorem 79) by which $e^{\pm ix} = \cos(x) \pm i\sin(x)$.
On the other hand, $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$.
[Recall that the first formula is an instance of $\operatorname{Re}(z) = \frac{1}{2}(z + \bar{z})$ and the second of $\operatorname{Im}(z) = \frac{1}{2i}(z - \bar{z})$.]

Example 81. Determine the general solution of $y'' - 4y' + 13y = 0$ using only real numbers.

Solution. The characteristic polynomial $p(D) = D^2 - 4D + 13$ has roots $2 + 3i, 2 - 3i$.

[We can use the quadratic formula to find these roots as $\frac{4 \pm \sqrt{4^2 - 4 \cdot 13}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$.]

Hence, the general solution in real form is $y(x) = C_1 e^{2x}\cos(3x) + C_2 e^{2x}\sin(3x)$.

Note. $e^{(2 \pm 3i)x} = e^{2x} e^{\pm 3ix} = e^{2x}(\cos(3x) \pm i\sin(3x))$

Armin Straub
straub@southalabama.edu

Review. A linear DE of order n is of the form

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x) y' + P_0(x) y = f(x).$$

The **general solution of linear DE** always takes the form

$$y(x) = y_p(x) + C_1 y_1(x) + \ldots + C_n\, y_n(x),$$

where y_p is any solution (called a **particular solution**) and $y_1, y_2, \ldots, y_n$ are solutions to the corresponding **homogeneous** linear DE.

- In terms of $D = \frac{\mathrm{d}}{\mathrm{d}x}$, the DE becomes: $Ly = f(x)$ with $L = D^n + P_{n-1}(x) D^{n-1} + \ldots + P_1(x) D + P_0(x)$.
- The inclusion of the $f(x)$ term makes $Ly = f(x)$ an **inhomogeneous** linear DE. The corresponding **homogeneous** DE is $Ly = 0$ (note that the zero function $y(x) = 0$ is a solution of $Ly = 0$).
- L is called a **linear differential operator**.
  - $L(C_1 y_1 + C_1 y_2) = C_1 L y_1 + C_2 L y_2$ **(linearity)**
    Comment. If you are familiar with linear algebra, think of L replaced with a matrix A and y_1, y_2 replaced with vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$. In that case, the same linearity property holds.
  - So, if y_1 solves $Ly = f(x)$, and y_2 solves $Ly = g(x)$, then $C_1 y_1 + C_2 y_2$ solves $C_1 f(x) + C_2\, g(x)$.
  - In particular, if y_1 and y_2 solve the homogeneous DE, then so does any linear combination $C_1 y_1 + C_2 y_2$. This explains why, for any homogeneous linear DE of order n, there are n solutions $y_1, y_2, \ldots, y_n$ such that the general solution is $y(x) = C_1 y_1(x) + \ldots + C_n\, y_n(x)$. Moreover, in that case, if we have a **particular solution** y_p of the inhomogeneous DE $Ly = f(x)$, then $y_p + C_1 y_1 + \ldots + C_n\, y_n$ is the general solution of $Ly = f(x)$.

Example 82. (review) Find the general solution of $y''' + 2y'' + y' = 0$.

Solution. The characteristic polynomial $p(D) = D(D+1)^2$ has roots $0, 1, 1$.
Hence, the general solution is $A + (B + Cx) e^x$.

Example 83. (review) Find the general solution of $y^{(7)} + 8y^{(6)} + 42y^{(5)} + 104y^{(4)} + 169y''' = 0$.

Use the fact that $-2 + 3i$ is a repeated characteristic root.

Solution. The characteristic polynomial $p(D) = D^3(D^2 + 4D + 13)^2$ has roots $0, 0, 0, -2 \pm 3i, -2 \pm 3i$.
[Since $-2 + 3i$ is a root so must be $-2 - 3i$. Repeating them once, together with $0, 0, 0$ results in 7 roots.]
Hence, the general solution is $(A + Bx + Cx^2) + (D + Ex)\, e^{-2x} \cos(3x) + (F + Gx) e^{-2x} \sin(3x)$.

Example 84. (preview) Determine the general solution of $y'' + 4y = 12x$. *Hint*: $3x$ is a solution.

Solution. Here, $p(D) = D^2 + 4$. Because of the hint, we know that a particular solution is $y_p = 3x$.
The homogeneous DE $p(D) y = 0$ has solutions $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$. [Make sure this is clear!]
Therefore, the general solution to the original DE is $y_p + C_1\, y_1 + C_2 y_2 = 3x + C_1 \cos(2x) + C_2 \sin(2x)$.

Just to make sure. The DE in operator notation is $Ly = f(x)$ with $L = D^2 + 4$ and $f(x) = 12x$.
Next. How to find the particular solution $y_p = 3x$ ourselves.

Armin Straub
straub@southalabama.edu

Inhomogeneous linear DEs: The method of undetermined coefficients

The method of undetermined coefficients allows us to solve any inhomogeneous linear DE $Ly = f(x)$ with constant coefficients if $f(x)$ is a polynomial times an exponential (or a linear combination of such terms).

More precisely, $Q(x)$ needs to be a solution of a homogeneous linear DE with constant coefficients.

Example 85. Determine the general solution of $y'' + 4y = 12x$.

Solution. The DE is $p(D)y = 12x$ with $p(D) = D^2 + 4$, which has roots $\pm 2i$. Thus, the general solution is $y(x) = y_p(x) + C_1\cos(2x) + C_2\sin(2x)$. It remains to find a particular solution y_p.

Since $D^2 \cdot (12x) = 0$, we apply D^2 to both sides of the DE to get the **homogeneous** DE $D^2(D^2 + 4) \cdot y = 0$.

Its general solution is $C_1 + C_2x + C_3\cos(2x) + C_4\sin(2x)$ and y_p must be of this form. Indeed, there must be a particular solution of the simpler form $y_p = C_1 + C_2x$ (because $C_3\cos(2x) + C_4\sin(2x)$ can be added to any y_p).

It remains to find appropriate values C_1, C_2 such that $y_p'' + 4y_p = 12x$. Since $y_p'' + 4y_p = 4C_1 + 4C_2x$, comparing coefficients yields $4C_1 = 0$ and $4C_2 = 12$, so that $C_1 = 0$ and $C_2 = 3$. In other words, $y_p = 3x$.

Therefore, the general solution to the original DE is $y(x) = 3x + C_1\cos(2x) + C_2\sin(2x)$.

Example 86. Determine the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. The DE is $p(D)y = e^{3x}$ with $p(D) = D^2 + 4D + 4 = (D + 2)^2$, which has roots $-2, -2$. Thus, the general solution is $y(x) = y_p(x) + (C_1 + C_2x)e^{-2x}$. It remains to find a particular solution y_p.

Since $(D - 3)e^{3x} = 0$, we apply $(D - 3)$ to the DE to get the **homogeneous** DE $(D - 3)(D + 2)^2y = 0$.

Its general solution is $(C_1 + C_2x)e^{-2x} + C_3e^{3x}$ and y_p must be of this form. Indeed, there must be a particular solution of the simpler form $y_p = Ce^{3x}$.

To determine the value of C, we plug into the original DE: $y_p'' + 4y_p' + 4y_p = (9 + 4 \cdot 3 + 4)Ce^{3x} \overset{!}{=} e^{3x}$. Hence, $C = 1/25$. Therefore, the general solution to the original DE is $y(x) = (C_1 + C_2x)e^{-2x} + \frac{1}{25}e^{3x}$.

We found a recipe for solving nonhomogeneous linear DEs with constant coefficients.

Our approach works for $p(D)y = f(x)$ whenever the right-hand side $f(x)$ is the solution of some homogeneous linear DE with constant coefficients: $q(D)f(x) = 0$

Theorem 87. (method of undetermined coefficients) To find a particular solution y_p to an inhomogeneous linear DE with constant coefficients $p(D)y = f(x)$:

- Find $q(D)$ so that $q(D)f(x) = 0$. [This does not work for all $f(x)$.]
- It follows that y_p solves the **homogeneous** DE $q(D)\,p(D)y = 0$.

  The characteristic polynomial of this DE has roots:

  - The roots $r_1, \ldots, r_n$ of the polynomial $p(D)$ (the "old" roots).
  - The roots $s_1, \ldots, s_m$ of the polynomial $q(D)$ (the "new" roots).
- Let $y_1^{\text{new}}, \ldots, y_m^{\text{new}}$ be the "new" solutions (i.e. not solutions of the "old" $p(D)y = 0$).

  We plug into $p(D)y_p = f(x)$ to find (unique) C_i so that $y_p = C_1y_1^{\text{new}} + \ldots + C_my_m^{\text{new}}$.

Because of the final step, this approach is often called **method of undetermined coefficients**.

For which $f(x)$ does this work? By Theorem 70, we know exactly which $f(x)$ are solutions to homogeneous linear DEs with constant coefficients: these are linear combinations of exponentials x^je^{rx} (which includes $x^j e^{ax}\cos(bx)$ and $x^j e^{ax}\sin(bx)$).

Armin Straub
straub@southalabama.edu

Example 88. (again) Determine the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. The "old" roots are $-2, -2$. The "new" roots are 3. Hence, there has to be a particular solution of the form $y_p = Ce^{3x}$. To find the value of C, we plug into the DE.

$y_p'' + 4y_p' + 4y_p = (9 + 4 \cdot 3 + 4)Ce^{3x} \overset{!}{=} e^{3x}$. Hence, $C = 1/25$.

Therefore, the general solution is $y(x) = \frac{1}{25}e^{3x} + (C_1 + C_2x)e^{-2x}$.

Example 89. Determine the general solution of $y'' + 4y' + 4y = 7e^{-2x}$.

Solution. The "old" roots are $-2, -2$. The "new" roots are -2. Hence, there has to be a particular solution of the form $y_p = Cx^2e^{-2x}$. To find the value of C, we plug into the DE.

$y_p' = C(-2x^2 + 2x)e^{-2x}$

$y_p'' = C(4x^2 - 8x + 2)e^{-2x}$

$y_p'' + 4y_p' + 4y_p = 2Ce^{-2x} \overset{!}{=} 7e^{-2x}$

It follows that $C = 7/2$, so that $y_p = \frac{7}{2}x^2e^{-2x}$. The general solution is $y(x) = \left(C_1 + C_2x + \frac{7}{2}x^2\right)e^{-2x}$.

Example 90. Consider the DE $y'' + 4y' + 4y = 2e^{3x} - 5e^{-2x}$.

(a) What is the simplest form (with undetermined coefficients) of a particular solution?

(b) Determine a particular solution using our results from Examples 88 and 89.

(c) Determine the general solution.

Solution.

(a) The "old" roots are $-2, -2$. The "new" roots are $3, -2$. Hence, there has to be a particular solution of the form $y_p = Ae^{3x} + Bx^2e^{-2x}$.

To find the (unique) values of A and B, we can plug into the DE. Alternatively, we can break the problem into two pieces as illustrated in the next part.

(b) Write the DE as $Ly = 2e^{3x} - 5e^{-2x}$ where $L = D^2 + 4D + 4$. In Example 88 we found that $y_1 = \frac{1}{25}e^{3x}$ satisfies $Ly_1 = e^{3x}$. Also, in Example 89 we found that $y_2 = \frac{7}{2}x^2e^{-2x}$ satisfies $Ly_2 = 7e^{-2x}$.

By linearity, it follows that $L(Ay_1 + By_2) = ALy_1 + BLy_2 = Ae^{3x} + 7Be^{-2x}$.

To get a particular solution y_p of our DE, we need $A = 2$ and $7B = -5$.

Hence, $y_p = 2y_1 - \frac{5}{7}y_2 = \frac{2}{25}e^{3x} - \frac{5}{2}x^2e^{-2x}$.

Comment. Of course, if we hadn't previously solved Examples 88 and 89, we could have plugged the result from the first part into the DE to determine the coefficients A and B. On the other hand, breaking the inhomogeneous part ($2e^{3x} - 5e^{-2x}$) up into pieces (here, e^{3x} and e^{-2x}) can help keep things organized, especially when working by hand.

(c) The general solution is $\frac{2}{25}e^{3x} - \frac{5}{2}x^2e^{-2x} + (C_1 + C_2x)e^{2x}$.

Example 91. Consider the DE $y'' - 2y' + y = 5\sin(3x)$.

(a) What is the simplest form (with undetermined coefficients) of a particular solution?

(b) Determine a particular solution.

(c) Determine the general solution.

Solution.

(a) Since $D^2 - 2D + 1 = (D-1)^2$, the "old" roots are $1, 1$. The "new" roots are $\pm 3i$. Hence, there has to be a particular solution of the form $y_p = A\cos(3x) + B\sin(3x)$.

(b) To find the values of A and B, we plug into the DE.

$y_p' = -3A\sin(3x) + 3B\cos(3x)$

$y_p'' = -9A\cos(3x) - 9B\sin(3x)$

$y_p'' - 2y_p' + y_p = (-8A - 6B)\cos(3x) + (6A - 8B)\sin(3x) \overset{!}{=} 5\sin(3x)$

Equating the coefficients of $\cos(x)$, $\sin(x)$, we obtain the two equations $-8A - 6B = 0$ and $6A - 8B = 5$.

Solving these, we find $A = \frac{3}{10}$, $B = -\frac{2}{5}$. Accordingly, a particular solution is $y_p = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x)$.

(c) The general solution is $y(x) = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x) + (C_1 + C_2x)e^x$.

Example 92. Consider the DE $y'' - 2y' + y = 5e^{2x}\sin(3x) + 7xe^x$. What is the simplest form (with undetermined coefficients) of a particular solution?

Solution. Since $D^2 - 2D + 1 = (D-1)^2$, the "old" roots are $1, 1$. The "new" roots are $2 \pm 3i, 1, 1$. Hence, there has to be a particular solution of the form $y_p = Ae^{2x}\cos(3x) + Be^{2x}\sin(3x) + Cx^2e^x + Dx^3e^x$.
(We can then plug into the DE to determine the (unique) values of the coefficients A, B, C, D.)

Example 93. (homework) What is the shape of a particular solution of $y'' + 4y' + 4y = x\cos(x)$?

Solution. The "old" roots are $-2, -2$. The "new" roots are $\pm i, \pm i$. Hence, there has to be a particular solution of the form $y_p = (C_1 + C_2 x)\cos(x) + (C_3 + C_4 x)\sin(x)$.

Continuing to find a particular solution. To find the value of the C_j's, we plug into the DE.

$y_p' = (C_2 + C_3 + C_4 x)\cos(x) + (C_4 - C_1 - C_2 x)\sin(x)$

$y_p'' = (2C_4 - C_1 - C_2 x)\cos(x) + (-2C_2 - C_3 - C_4 x)\sin(x)$

$$y_p'' + 4y_p' + 4y_p = (3C_1 + 4C_2 + 4C_3 + 2C_4 + (3C_2 + 4C_4)x)\cos(x) + (-4C_1 - 2C_2 + 3C_3 + 4C_4 + (-4C_2 + 3C_4)x)\sin(x) \overset{!}{=} x\cos(x).$$

Equating the coefficients of $\cos(x)$, $x\cos(x)$, $\sin(x)$, $x\sin(x)$, we get the equations $3C_1 + 4C_2 + 4C_3 + 2C_4 = 0$, $3C_2 + 4C_4 = 1$, $-4C_1 - 2C_2 + 3C_3 + 4C_4 = 0$, $-4C_2 + 3C_4 = 0$.

Solving (this is tedious!), we find $C_1 = -\frac{4}{125}$, $C_2 = \frac{3}{25}$, $C_3 = -\frac{22}{125}$, $C_4 = \frac{4}{25}$.

Hence, $y_p = \left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)$.

Example 94. (homework) What is the shape of a particular solution of $y'' + 4y' + 4y = 4e^{3x}\sin(2x) - x\sin(x)$.

Solution. The "old" roots are $-2, -2$. The "new" roots are $3 \pm 2i, \pm i, \pm i$.

Hence, there has to be a particular solution of the form

$y_p = C_1 e^{3x}\cos(2x) + C_2 e^{3x}\sin(2x) + (C_3 + C_4 x)\cos(x) + (C_5 + C_6 x)\sin(x)$.

Continuing to find a particular solution. To find the values of $C_1, \ldots, C_6$, we plug into the DE. But this final step is so boring that we don't go through it here. Computers (currently?) cannot afford to be as selective; mine obediently calculated: $y_p = -\frac{4}{841}e^{3x}(20\cos(2x) - 21\sin(2x)) + \frac{1}{125}((-22 + 20x)\cos(x) + (4 - 15x)\sin(x))$

A more general method for finding particular solutions: variation of parameters

The method of undetermined coefficients allows us to solve an inhomogeneous linear DE $Ly = f(x)$ for certain functions $f(x)$. The next method has no restriction on the functions $f(x)$ we can handle. The price to pay for this is that the method is usually more laborious.

Review. To find the general solution of an inhomogeneous linear DE $Ly = f(x)$, we only need to find a single **particular solution** y_p. Then the general solution is $y_p + y_h$, where y_h is the general solution of $Ly = 0$.

Theorem 95. (variation of parameters) A particular solution to the inhomogeneous second-order linear DE $Ly = y'' + P_1(x)y' + P_0(x)y = f(x)$ is given by:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x), \quad u_1(x) = -\int \frac{y_2(x)f(x)}{W(x)}\mathrm{d}x, \quad u_2(x) = \int \frac{y_1(x)f(x)}{W(x)}\mathrm{d}x,$$

where y_1, y_2 are independent solutions of $Ly = 0$ and $W = y_1 y_2' - y_1' y_2$ is their Wronskian.

Comment. We obtain the general solution if we consider all possible constants of integration in the formula for y_p.

Armin Straub
straub@southalabama.edu

Proof. Let us look for a particular solution of the form $y_p = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$.

This "ansatz" is called **variation of constants/parameters**. We plug into the DE to determine conditions on u_1, u_2 so that y_p is a solution. The DE will give us one condition and (since there are two unknowns), it is reasonable to expect that we can impose a second condition (labelled below as "our wish") to make our life simpler.

We compute $y_p' = \underbrace{u_1' y_1 + u_2' y_2}_{=0 \text{ (our wish)}} + u_1 y_1' + u_2 y_2'$ and, thus, $y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$.

["Our wish" was chosen so that y_p'' would only involve first derivatives of u_1 and u_2.]

Therefore, plugging into the DE results in

$$L y_p = u_1' y_1' + u_2' y_2' + \underbrace{u_1 y_1'' + u_2 y_2'' + P_1(x)(u_1 y_1' + u_2 y_2') + P_0(x)(u_1 y_1 + u_2 y_2)}_{= u_1 L y_1 + u_2 L y_2 = 0} \overset{!}{=} f(x).$$

We conclude that y_p solves the DE if the following two conditions (the first is "our wish") are satisfied:

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0, \\ u_1' y_1' + u_2' y_2' &= f(x). \end{aligned}$$

These are linear equations in u_1' and u_2'. Solving gives $u_1' = \frac{-y_2 f(x)}{y_1 y_2' - y_1' y_2}$ and $u_2' = \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2}$, and it only remains to integrate. □

Comment. In matrix-vector form, the equations are $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$.

Our solution then follows from multiplying $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} = \frac{1}{y_1 y_2' - y_1' y_2} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}$ with $\begin{bmatrix} 0 \\ f(x) \end{bmatrix}$.

Advanced comment. $W = y_1 y_2' - y_1' y_2$ is called the **Wronskian** of y_1 and y_2. In general, given a linear homogeneous DE of order n with solutions y_1, ..., y_n, the Wronskian of y_1, ..., y_n is the determinant of the matrix where each column consists of the derivatives of one of the y_i. One useful property of the Wronskian is that it is nonzero if and only if the $y_1, \ldots, y_n$ are linearly independent and therefore generate the general solution.

Example 96. Determine the general solution of $y'' - 2y' + y = \frac{e^x}{x}$.

Solution. This DE is of the form $Ly = f(x)$ with $L = D^2 - 2D + 1$ and $f(x) = \frac{e^x}{x}$.

Since $L = (D-1)^2$, the homogeneous DE has the two solutions $y_1 = e^x$, $y_2 = x e^x$.

The corresponding Wronskian is $W = y_1 y_2' - y_1' y_2 = e^x (1+x) e^x - e^x (x e^x) = e^{2x}$.

By variation of parameters (Theorem 95), we find that a particular solution is

$$y_p = -y_1 \int \frac{y_2 f}{W} \mathrm{d}x + y_2 \int \frac{y_1 f}{W} \mathrm{d}x = -e^x \int 1 \mathrm{d}x + x e^x \int \frac{1}{x} \mathrm{d}x = x e^x (\ln|x| - 1).$$

The general solution therefore is $x e^x (\ln|x| - 1) + (C_1 + C_2 x) e^x$.

If we prefer, a simplified particular solution is $x e^x \ln|x|$ (because we can add any multiple of $x e^x$ to y_p). Then the general solution takes the simplified form $x e^x \ln|x| + (C_1 + C_2 x) e^x$.

Comment. Adding constants of integration in the formula for y_p, we get $-e^x (x + D_1) + x e^x (\ln|x| + D_2)$, which is the general solution. Any choice of constants suffices to give us a particular solution.

Important comment. Note that we cannot use the method of undetermined coefficients here because the inhomogeneous term $f(x) = \frac{e^x}{x}$ is not of the appropriate form. See the next example for a case where both methods can be applied.

Example 97. (homework) Determine the general solution of $y'' + 4y' + 4y = e^{3x}$.

(a) Using the method of undetermined coefficients.

(b) Using variation of constants.

Solution.

(a) We already did this in Example 88: The "old" roots are $-2, -2$. The "new" roots are 3. Hence, there has to be a particular solution of the form $y_p = Ce^{3x}$. To find the value of C, we plug into the DE.

$y_p'' + 4y_p' + 4y_p = (9 + 4\cdot 3 + 4)Ce^{3x} \stackrel{!}{=} e^{3x}$. Hence, $C = 1/25$.

Therefore, the general solution is $y(x) = \frac{1}{25}e^{3x} + (C_1 + C_2x)e^{-2x}$.

(b) This DE is of the form $Ly = f(x)$ with $L = D^2 + 4D + 4$ and $f(x) = e^{3x}$.

Since $L = (D+2)^2$, the homogeneous DE has the two solutions $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$.

The corresponding Wronskian is $W = y_1y_2' - y_1'y_2 = e^{-2x}(1-2x)e^{-2x} - (-2e^{-2x})xe^{-2x} = e^{-4x}$.

By variation of parameters (Theorem 95), we find that a particular solution is

$$\begin{aligned} y_p &= -y_1\int \frac{y_2 f}{W}\mathrm{d}x + y_2\int\frac{y_1 f}{W}\mathrm{d}x \\ &= -e^{-2x}\underbrace{\int xe^{5x}\mathrm{d}x}_{=\frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x}} + xe^{-2x}\underbrace{\int e^{5x}\mathrm{d}x}_{=\frac{1}{5}e^{5x}} = \frac{1}{25}e^{3x}. \end{aligned}$$

The general solution therefore is $\frac{1}{25}e^{3x} + (C_1 + C_2x)e^{-2x}$, which matches what we got before.

Example 98. (homework) Determine the general solution of $y'' + 4y' + 4y = 7e^{-2x}$.

(a) Using the method of undetermined coefficients.

(b) Using variation of constants.

Solution.

(a) We already did this in Example 89: The "old" roots are $-2, -2$. The "new" roots are -2. Hence, there has to be a particular solution of the form $y_p = Cx^2e^{-2x}$. To find the value of C, we plug into the DE.

$y_p' = C(-2x^2 + 2x)e^{-2x}$

$y_p'' = C(4x^2 - 8x + 2)e^{-2x}$

$y_p'' + 4y_p' + 4y_p = 2Ce^{-2x} \stackrel{!}{=} 7e^{-2x}$

It follows that $C = 7/2$, so that $y_p = \frac{7}{2}x^2e^{-2x}$. The general solution is $y(x) = \left(C_1 + C_2x + \frac{7}{2}x^2\right)e^{-2x}$.

(b) This DE is of the form $Ly = f(x)$ with $L = D^2 + 4D + 4$ and $f(x) = 7e^{-2x}$.

Since $L = (D+2)^2$, the homogeneous DE has the two solutions $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$.

The corresponding Wronskian is $W = y_1y_2' - y_1'y_2 = e^{-2x}(1-2x)e^{-2x} - (-2e^{-2x})xe^{-2x} = e^{-4x}$.

By variation of parameters (Theorem 95), we find that a particular solution is

$$\begin{aligned} y_p &= -y_1\int \frac{y_2 f}{W}\mathrm{d}x + y_2\int\frac{y_1 f}{W}\mathrm{d}x \\ &= -e^{-2x}\underbrace{\int 7x\mathrm{d}x}_{=\frac{7}{2}x^2} + xe^{-2x}\underbrace{\int 7\mathrm{d}x}_{=7x} = \frac{7}{2}x^2e^{-2x}. \end{aligned}$$

The general solution therefore is $\frac{7}{2}x^2e^{-2x} + (C_1 + C_2x)e^{-2x}$, which matches what we got before.

Review. The method of undetermined coefficients versus variation of parameters.

A closer look at second-order linear DEs

Application: motion of a mass on a spring

Example 99. The motion of a mass m attached to a spring is described by

$$my'' + ky = 0$$

where y is the displacement from the equilibrium position and $k > 0$ is the spring constant.

Why? This follows from Hooke's law $F = -ky$ combined with Newton's second law $F = ma = my''$. (Note that the minus sign is needed because the force on the mass is in direction opposite to the displacement.)

Comment. By measuring y as the displacement from equilibrium, it doesn't matter whether the mass is attached horizontally or vertically (gravity is taken into account by the extra stretch in the spring due to the mass).

Solving this DE, we find that the general solution is

$$y(t) = A\cos(\omega t) + B\sin(\omega t)$$

where $\omega = \sqrt{k/m}$ (note that the characteristic roots are $\pm i\sqrt{\frac{k}{m}}$). We observe that:

- The motion $y(t)$ is periodic with **period** $2\pi/\omega$.

  This follows from the fact that both $\cos(t)$ and $\sin(t)$ have period 2π.

- The **amplitude** of the motion $y(t)$ is $\sqrt{A^2 + B^2}$.

  This follows from the fact that $y(t) = A\cos(\omega t) + B\sin(\omega t) = r\cos(\omega t - \alpha)$ where (r, α) are the **polar coordinates** for (A, B). In particular, the amplitude is $r = \sqrt{A^2 + B^2}$.

  Can you explain the reason for being able to write $y(t)$ as $r\cos(\omega t - \alpha)$ using DEs? More on this next time...

Example 100. What is the period and the amplitude of the oscillations $\cos(4t) - 3\sin(4t)$?

Solution. The period is $\frac{2\pi}{4} = \frac{\pi}{2}$.

The amplitude is $\sqrt{1^2 + (-3)^2} = \sqrt{10}$.

The amplitude of oscillations

Example 101. If (r,α) are the **polar coordinates** for (A,B), then

$$A\cos(\omega t)+B\sin(\omega t)=r\cos(\omega t-\alpha).$$

In particular, $A\cos(\omega t)+B\sin(\omega t)$ is periodic with **amplitude** $r=\sqrt{A^2+B^2}$.

ω is the (circular) **frequency** and α is called the **phase angle**.

Why? Both sides solve

$$y''+\omega y=0.$$

The LHS has initial values $y(0)=A$ and $y'(0)=\omega B$, the RHS has $y(0)=r\cos(\alpha)$ and $y'(0)=r\omega\sin(\alpha)$. Hence, the two are equal if $A=r\cos(\alpha)$ and $B=r\sin(\alpha)$.

Alternatively. If you like trig identities, this follows from:
$A\cos(\omega t)+B\sin(\omega t)=r(\cos(\alpha)\cos(\omega t)+\sin(\alpha)\sin(\omega t))=r\cos(\omega t-\alpha)$.

Review. How to calculate the polar coordinates (r,α) for (A,B)?

We need to find $r\geqslant 0$ and $\alpha\in[0,2\pi)$ such that $(A,B)=r(\cos\alpha,\sin\alpha)$.

Hence, $r=\sqrt{A^2+B^2}$ and α is determined by $\cos(\alpha)=\frac{A}{r}$ and $\sin(\alpha)=\frac{B}{r}$.

In particular, $\tan(\alpha)=\frac{B}{A}$ and, if careful, we can compute α using $\tan^{-1}$ as

$$\alpha=\tan^{-1}\left(\frac{B}{A}\right)+\begin{cases}0, & \text{if } (A,B) \text{ in first quadrant,}\\ 2\pi, & \text{if } (A,B) \text{ in fourth quadrant,}\\ \pi, & \text{otherwise.}\end{cases}$$

Example 102. (again) What is the period and the amplitude of the oscillations $\cos(4t)-3\sin(4t)$?

Solution. The period is $\frac{2\pi}{4}=\frac{\pi}{2}$.

The amplitude is $\sqrt{1^2+(-3)^2}=\sqrt{10}$.

Example 103. The motion of a mass on a spring is described by $5y''+2y=0$, $y(0)=3$, $y'(0)=-1$. What is the period and the amplitude of the resulting oscillations?

Solution. The characteristic roots are $\pm i\omega$ with $\omega=\sqrt{\frac{2}{5}}$. The general solution is $y(t)=A\cos(\omega t)+B\sin(\omega t)$.

The period of the oscillations therefore is $\frac{2\pi}{\omega}=2\pi\sqrt{\frac{5}{2}}=\pi\sqrt{10}$.

To meet the initial conditions, we need $y(0)=A\overset{!}{=}3$ and $y'(0)=\omega B\overset{!}{=}-1$. The latter implies $B=-\frac{1}{\omega}=-\sqrt{\frac{5}{2}}$.

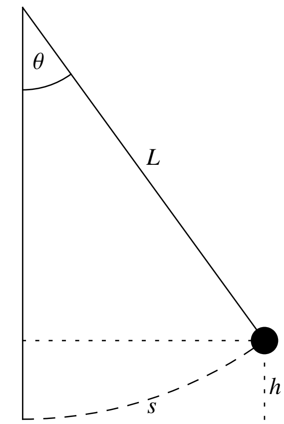
Hence, the amplitude of the oscillations is $\sqrt{A^2+B^2}=\sqrt{3^2+\frac{5}{2}}=\sqrt{\frac{23}{2}}$.

Example 104. Show that the motion of an ideal pendulum is described by

$$L\theta'' + g\sin(\theta) = 0,$$

where θ is the angular displacement and L is the length of the pendulum.

And, as usual, g is acceleration due to gravity.



For short times and small angles, this motion is approximately described by

$$L\theta'' + g\theta = 0.$$

This is because, if θ is small, then $\sin(\theta) \approx \theta$. For instance, for $\theta = 15°$ the error $\theta - \sin\theta$ is about 1%.

Solution. (Newton's second law) The tangential component of the gravitational force is $F = -\sin\theta \cdot mg$. Combining this with Newton's second law, according to which $F = ma = mL\theta''$ (note that $a = s''$ where $s = L\theta$), we obtain the claimed DE.

Solution. (conservation of energy) Alternatively, we can use conservation of energy to derive the DE. Again, we assume the string to be massless, and let m be the swinging mass. Let s and h be as in the sketch above.

The velocity (more accurately, the speed) of the mass is $v = \frac{\mathrm{d}s}{\mathrm{d}t} = L\frac{\mathrm{d}\theta}{\mathrm{d}t}$.

Its kinetic energy therefore is $T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$.

On the other hand, the potential energy is $V = mgh = mgL(1 - \cos\theta)$ (weight mg times height h).

By the principle of conservation of energy, the sum of these is constant: $T + V = \mathrm{const}$

Taking the time derivative, this becomes $\frac{1}{2}mL^2 2\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + mgL\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$. Cancelling terms, we obtain the DE.

Example 105. The motion of a pendulum is described by $\theta'' + 9\theta = 0$, $\theta(0) = 1/4$, $\theta'(0) = 0$. What is the period and the amplitude of the resulting oscillations?

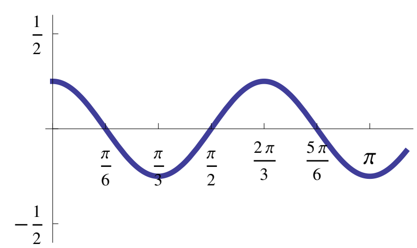
Solution. The roots of the characteristic polynomial are $\pm 3i$.

Hence, $\theta(t) = A\cos(3t) + B\sin(3t)$. $\theta(0) = A = 1/4$. $\theta'(0) = 3B = 0$.

Therefore, the solution is $\theta(t) = 1/4\cos(3t)$.

Hence, the period is $2\pi/3$ and the amplitude is $1/4$.

Comment. $1/4$ is about $14.3°$.



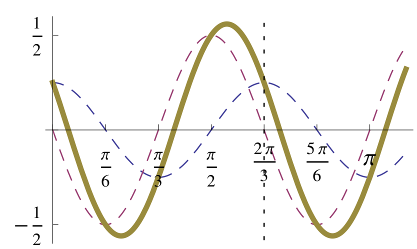
Example 106. The motion of a pendulum is described by $\theta'' + 9\theta = 0$, $\theta(0) = 1/4$, $\theta'(0) = -\frac{3}{2}$ ("initial kick"). What is the period and the amplitude of the resulting oscillations?

Solution. This time, $\theta(0) = A = 1/4$. $\theta'(0) = 3B = -3/2$.

Therefore, the solution is $\theta(t) = \frac{1}{4}\cos(3t) - \frac{1}{2}\sin(3t)$.

Hence, the period is $2\pi/3$ and the amplitude is $\sqrt{\frac{1}{4^2} + \frac{1}{2^2}} = \frac{\sqrt{5}}{4} \approx 0.559$.

Comment. Using polar coordinates, we get $\theta(t) = \frac{\sqrt{5}}{4}\cos(3t - \alpha)$ with phase angle $\alpha = \tan^{-1}(-2) + 2\pi \approx 5.176$.



Armin Straub
straub@southalabama.edu

The qualitative effects of damping

The motion of a mass on a spring (or the approximate motion of a pendulum), with damping taken into account, can be modeled by the DE

$$y'' + dy' + cy = 0$$

with $c > 0$ and $d \geqslant 0$. The term dy' models damping (e.g. friction, air resistance) proportional to the velocity y'.

The characteristic equation $r^2 + dr + c = 0$ has roots $\frac{1}{2}\left(-d \pm \sqrt{d^2 - 4c}\right)$.

The nature of the solutions depends on whether the **discriminant** $\Delta = d^2 - 4c$ is positive, negative, or zero.

Undamped. $d = 0$. In that case, $\Delta < 0$. We get two complex roots $\pm i\omega$ with $\omega = \sqrt{c}$.

Solutions: $A\cos(\omega t) + B\sin(\omega t) = r\cos(\omega t - \alpha)$ where $(A, B) = r(\cos\alpha, \sin\alpha)$
These are oscillations with frequency ω and amplitude r.

Underdamped. $d > 0$, $\Delta < 0$. We get two complex roots $-\rho \pm i\omega$ with $-\rho = -d/2 < 0$.

Solutions: $e^{-\rho t}[A\cos(\omega t) + B\sin(\omega t)] = e^{-\rho t}[r\cos(\omega t - \alpha)]$ ($\rightarrow 0$ as $t \rightarrow \infty$)
These are oscillations with amplitude going to zero.

Critically damped. $d > 0$, $\Delta = 0$. We get one (double) real root $-\rho < 0$.

Solutions: $(A + Bt)\, e^{-\rho t}$ ($\rightarrow 0$ as $t \rightarrow \infty$)
There are no oscillations. (Can you see why we cross the t-axis at most once?)

Overdamped. $d > 0$, $\Delta > 0$. We get two real roots $-\rho_1, -\rho_2 < 0$. [negative because $c, d > 0$]

Solutions: $A e^{-\rho_1 t} + B e^{-\rho_2 t}$ ($\rightarrow 0$ as $t \rightarrow \infty$)
There are no oscillations. (Again, there is at most one crossing of the t-axis.)

Example 107. The motion of a mass on a spring is described by $5y'' + dy' + 2y = 0$ with $d > 0$. For which value of d is the motion critically damped? Underdamped? Overdamped?

Solution. The characteristic roots are $\frac{1}{2}\left(-d \pm \sqrt{d^2 - 40}\right)$. The motion is critically damped if $d^2 - 40 = 0$. Equivalently, the motion is critically damped $d = \sqrt{40}$.

Consequently, the motion is underdamped if $d < \sqrt{40}$ (then we get complex roots and the solutions will involve oscillations), and it is overdamped if $d > \sqrt{40}$ (the roots are real and the solutions will not involve oscillations).

Example 108. The motion of a mass on a spring is described by $my'' + 3y' + 2y = 0$. For which value of m is the motion critically damped? Underdamped? Overdamped?

Solution. The characteristic roots are $\frac{1}{2}(-3 \pm \sqrt{9 - 8m})$. The motion is critically damped if $9 - 8m = 0$. Equivalently, the motion is critically damped $m = \frac{9}{8}$.

Consequently, the motion is underdamped if $m > \frac{9}{8}$ (then we get complex roots and the solutions will involve oscillations), and it is overdamped if $m < \frac{9}{8}$ (the roots are real and the solutions will not involve oscillations).

Adding external forces and the phenomenon of resonance

The motion of a mass m on a spring, with damping and with an external force $f(t)$ taken into account, can be modeled by the DE

$$my'' + dy' + ky = f(t).$$

Note that each term is representing a force: $my'' = ma$ is the force due to Newton's second law ($F = ma$), the term dy' models damping (proportional to the velocity), the term ky represents the force due to Hooke's law, and the term $f(t)$ represents an external force that acts on the mass at time t.

Example 109. Describe the solutions of $y'' + 4y = \cos(\lambda t)$. (Here, $\lambda > 0$ is a constant.)

Solution. The "old" roots are $\pm 2i$ so that 2 is the **natural frequency** (the frequency at which the system would oscillate in the absence of external forces; mathematically, this reflects the fact that the general solution to the corresponding homogeneous DE is $A\cos(2t) + B\sin(2t)$, which has frequency $\omega = 2$).

The "new" roots are $\pm\lambda i$ where λ is the **external frequency**.

Case 1: $\boldsymbol{\lambda \neq 2}$. Then there is a particular solution of the form $y_p = A\cos(\lambda t) + B\sin(\lambda t)$. To determine the unique values of A, B, we plug into the DE:

$$y_p'' + 4y_p = (4 - \lambda^2)A\cos(\lambda t) + (4 - \lambda^2)B\sin(\lambda t) \overset{!}{=} \cos(\lambda t)$$

We conclude that $(4 - \lambda^2)A = 1$ and $(4 - \lambda^2)B = 0$. Solving these, we find $A = 1/(4 - \lambda^2)$ and $B = 0$.

Thus, the general solution is of the form $y = \frac{1}{4 - \lambda^2}\cos(\lambda t) + C_1\cos(2t) + C_2\sin(2t)$.

Case 2: $\boldsymbol{\lambda = 2}$. Now, there is a particular solution of the form $y_p = At\cos(2t) + Bt\sin(2t)$. To determine the unique values of A, B, we again plug into the DE (which is more work this time):

$$y_p'' + 4y_p \overset{\text{work}}{=} 4B\cos(2t) - 4A\sin(2t) \overset{!}{=} \cos(2t)$$

We conclude that $4B = 1$ and $-4A = 0$. Solving these, we find $A = 0$ and $B = 1/4$.

Thus, the general solution is of the form $y = \frac{1}{4}t\sin(2t) + C_1\cos(2t) + C_2\sin(2t)$.

Note that the amplitude in y_p increases without bound (so that the same is true for the general solution).

This phenomenon is called **resonance**; it occurs if an external frequency matches a natural frequency.

If an external frequency matches a natural frequency, then **resonance** occurs.

In that case, we obtain amplitudes that grow without bound.

Resonance (or anything close to it) is very important for practical purposes because large amplitudes can be very destructive: singing to shatter glass, earth quake waves and buildings, marching soldiers on bridges, ...

Comment. Mathematically speaking, the "old" and "new" roots overlap in an inhomogeneous linear DE. In that case, the solutions acquire a factor of the variable t (or x) which changes the nature of the solutions (and results in unbounded amplitudes).

Example 110. Consider $y'' + 9y = 10\cos(2\lambda t)$. For what value of λ does resonance occur?

Solution. The natural frequency is 3. The external frequency is 2λ. Hence, resonance occurs when $\lambda = \frac{3}{2}$.

Armin Straub
straub@southalabama.edu

Example 111. The motion of a mass on a spring under an external force is described by $5y'' + 2y = -2\sin(3\lambda t)$. For which value of λ does resonance occur?

Solution. The natural frequency is $\sqrt{\frac{2}{5}}$. The external frequency is 3λ. Hence, resonance occurs when $\lambda = \frac{1}{3}\sqrt{\frac{2}{5}}$.

Example 112. The motion of a mass on a spring under an external force is described by $3y'' + ry = \cos(t/2)$. For which value of $r > 0$ does resonance occur?

Solution. The natural frequency is $\sqrt{\frac{r}{3}}$. The external frequency is $\frac{1}{2}$. Hence, resonance occurs when $\sqrt{\frac{r}{3}} = \frac{1}{2}$. This happens if $r = 3 \cdot \left(\frac{1}{2}\right)^2 = \frac{3}{4}$.

External forces plus damping

In the presence of both damping and a periodic external force, the motion $y(t) = y_{\text{tr}} + y_{\text{sp}}$ of a mass on a spring splits into **transient motion** y_{tr} (with $y_{\text{tr}}(t) \to 0$ as $t \to \infty$) and **steady periodic oscillations** y_{sp}. The following example spells this out.

Comment. Note that y_{sp} will correspond to the simplest particular solution, while y_{tr} corresponds to the solution of the corresponding homogeneous system (where we have no external force).

Example 113. A forced mechanical oscillator is described by $2y'' + 2y' + y = 10\sin(t)$. As $t \to \infty$, what are the period and the amplitude of the resulting steady periodic oscillations?

Solution. The "old" roots are $\frac{1}{4}(-2 \pm \sqrt{4-8}) = -\frac{1}{2} \pm \frac{1}{2}i$. Accordingly, the system without external force is underdamped (because of the $\pm i/2$ the solutions will involve oscillations).

The "new" roots are $\pm i$ so that there must be a particular solution $y_p = A\cos(t) + B\sin(t)$ with coefficients A, B that we need to determine by plugging into the DE. This results in $A = -4$ and $B = -2$ (do it!).

Hence, the general solution is $y(t) = \underbrace{-4\cos(t) - 2\sin(t)}_{y_{\text{sp}}} + \underbrace{e^{-t/2}\left(C_1\cos\left(\frac{t}{2}\right) + C_2\sin\left(\frac{t}{2}\right)\right)}_{y_{\text{tr}} \to 0 \text{ as } t \to \infty}$.

The period of $y_{\text{sp}} = -4\cos(t) - 2\sin(t)$ is 2π and the amplitude is $\sqrt{(-4)^2 + (-2)^2} = \sqrt{20}$.

Comment. Using the polar coordinates $(-4, -2) = \sqrt{20}(\cos\alpha, \sin\alpha)$ where $\alpha = \tan^{-1}(1/2) + \pi \approx 3.605$, we can alternatively express the steady periodic oscillations as $y_{\text{sp}} = -4\cos(t) - 2\sin(t) = \sqrt{20}(\cos(t - \alpha))$.

Example 114. A forced mechanical oscillator is described by $y'' + 5y' + 6y = 2\cos(3t)$. What are the (circular) frequency and the amplitude of the resulting steady periodic oscillations?

Solution. The "old" roots are $-2, -3$. Accordingly, the system without external force is overdamped (the solutions will not involve oscillations).

The "new" roots are $\pm 3i$ so that there must be a particular solution $y_p = A\cos(3t) + B\sin(3t)$ with coefficients A, B that we need to determine by plugging into the DE. To do so, we compute $y_p' = -3A\sin(3t) + 3B\cos(3t)$ as well as $y_p'' = -9A\cos(3t) - 9B\sin(3t)$.

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= (-9A\cos(3t) - 9B\sin(3t)) + 5(-3A\sin(3t) + 3B\cos(3t)) + 6(A\cos(3t) + B\sin(3t)) \\ &= (-9A + 15B + 6A)\cos(3t) + (-9B - 15A + 6B)\sin(3t) \\ &\overset{!}{=} 2\cos(3t) \end{aligned}$$

This results in the two equations $-3A + 15B = 2$ and $-3B - 15A = 0$, which we solve to find $A = -\frac{1}{39}$ and $B = \frac{5}{39}$.

The general solution is $y(t) = \underbrace{-\frac{1}{39}\cos(3t) + \frac{5}{39}\sin(3t)}_{y_{\text{sp}}} + \underbrace{C_1e^{-2t} + C_2e^{-3t}}_{y_{\text{tr}} \to 0 \text{ as } t \to \infty}$.

The frequency of $y_{\text{sp}} = -\frac{1}{39}\cos(3t) + \frac{5}{39}\sin(3t)$ is 3 and the amplitude is $\sqrt{\left(-\frac{1}{39}\right)^2 + \left(\frac{5}{39}\right)^2} = \sqrt{\frac{2}{117}}$.

Armin Straub
straub@southalabama.edu

Example 115. Find the steady periodic solution to $y'' + 2y' + 5y = \cos(\lambda t)$. What is the amplitude of the steady periodic oscillations? For which λ is the amplitude maximal?

Solution. The "old" roots are $-1 \pm 2i$.

[Not really needed, because positive damping prevents duplication; can you see it?]

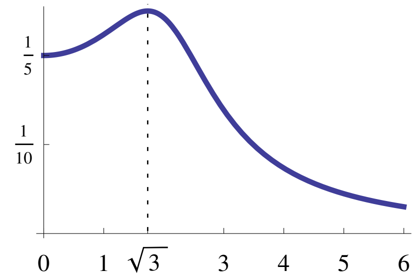
Hence, $y_{\text{sp}} = A\cos(\lambda t) + B\sin(\lambda t)$ and to find A, B we need to plug into the DE.

Doing so, we find $A = \frac{5-\lambda^2}{(5-\lambda^2)^2 + 4\lambda^2}$, $B = \frac{2\lambda}{(5-\lambda^2)^2 + 4\lambda^2}$.

Thus, the amplitude of y_{sp} is $r(\lambda) = \sqrt{A^2 + B^2} = \frac{1}{\sqrt{(5-\lambda^2)^2 + 4\lambda^2}}$.

The function $r(\lambda)$ is sketched to the right. It has a maximum at $\lambda = \sqrt{3}$ at which the amplitude is unusually large (well, here it is not very pronounced). We say that **practical resonance** occurs for $\lambda = \sqrt{3}$.

[For comparison, without damping, (pure) resonance occurs for $\lambda = \sqrt{5}$.]



Example 116. (homework) A car is going at constant speed v on a washboard road surface ("bumpy road") with height profile $y(s) = a\cos\left(\frac{2\pi s}{L}\right)$. Assume that the car oscillates vertically as if on a spring (no dashpot). Describe the resulting oscillations.

Solution. With x as in the sketch, the spring is stretched by $x - y$. Hence, by Hooke's and Newton's laws, its motion is described by $mx'' = -k(x - y)$.

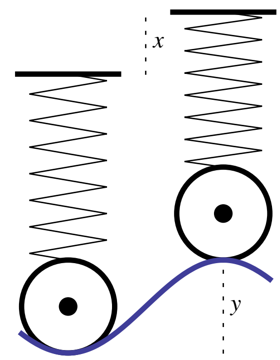
At constant speed, $s = vt$ and we obtain the DE $mx'' + kx = ky = ka\cos\left(\frac{2\pi vt}{L}\right)$, which is inhomogeneous linear with constant coefficients. Let's solve it.

"Old" roots: $\pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$. $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency.

"New" roots: $i\frac{2\pi v}{L} = \pm i\omega$. $\omega = \frac{2\pi v}{L}$ is the external frequency.

Case 1: $\omega \neq \omega_0$. Then a particular solution is $x_p = b_1\cos(\omega t) + b_2\sin(\omega t) = A\cos(\omega t - \alpha)$ for unique values of b_1, b_2 (which we do not compute here). The general solution is of the form $x = x_p + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$.

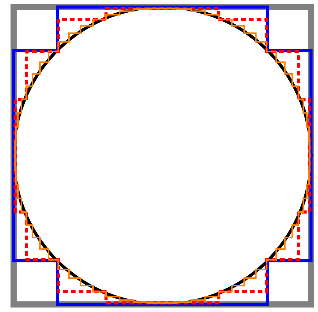
Case 2: $\omega = \omega_0$. Then a particular solution is $x_p = t[b_1\cos(\omega t) + b_2\sin(\omega t)] = At\cos(\omega t - \alpha)$ for unique values of b_1, b_2 (which we do not compute). Note that the amplitude in x_p increases without bound; the same is true for the general solution $x = x_p + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$. This phenomenon is called resonance; it occurs if an external frequency matches a natural frequency.



The first "car" is assumed to be in equilibrium.

Armin Straub
straub@southalabama.edu

(A Halloween scare!) π is the perimeter of a circle enclosed in a square with edge length 1. The perimeter of the square is 4, which approximates π. To get a better approximation, we "fold" the vertices of the square towards the circle (and get the blue polygon). This construction can be repeated for even better approximations and, in the limit, our shape will converge to the true circle. At each step, the perimeter is 4, so we conclude that $\pi = 4$, contrary to popular belief.



Can you pin-point the fallacy in this argument?

(We are not doing something completely silly! For instance, the areas of our approximations do converge to $\pi / 4$, the area of the circle.)

The "solution" is below...

($\pi = 4$, "solution")

We are constructing curves c_n with the property that $c_n \to c$ where c is the circle. This convergence can be understood, for instance, in the same sense $\|c_n - c\| \to 0$ with the norm measuring the maximum distance between the two curves.

Since $c_n \to c$ we then wanted to conclude that $\text{perimeter}(c_n) \to \text{perimeter}(c)$, leading to $4 \to \pi$.

However, in order to conclude from $x_n \to x$ that $f(x_n) \to f(x)$ we need that f is continuous (at x)!!

The "function" perimeter, however, is not continuous. In words, this means that (as we see in this example) curves can be arbitrarily close, yet have very different arc length.

We can dig a little deeper: as we learned in Calculus II, the arc length of a function $y = f_n(x)$ for $x \in [a, b]$ is

$$\int_a^b \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \int_a^b \sqrt{1 + f_n'(x)^2}\,\mathrm{d}x.$$

Observe that this involves $f_n'(x)$. Try to see why the operator D that sends f to f' is not continuous with respect to the distance induced by the norm

$$\|f\| = \left(\int_a^b f(x)^2 \mathrm{d}x\right)^{1/2}.$$

In words, two functions f and g can be arbitrarily close, yet have very different derivatives f' and g'.

That's a huge issue in **functional analysis**, which is the generalization of linear algebra to infinite dimensional spaces (like the space of all differentiable functions). The linear operators ("matrices") on these spaces frequently fail to be continuous.

Systems of differential equations

Modeling two connected fluid tanks

Example 117. Consider two brine tanks. Tank T_1 contains 24gal water containing 3lb salt, and tank T_2 contains 9gal pure water.

- T_1 is being filled with 54gal/min water containing 0.5lb/gal salt.
- 72gal/min well-mixed solution flows out of T_1 into T_2.
- 18gal/min well-mixed solution flows out of T_2 into T_1.
- Finally, 54gal/min well-mixed solution is leaving T_2.

How much salt is in the tanks after t minutes?

Solution. Note that the amount of water in each tank is constant because the flows balance each other. Let $y_i(t)$ denote the amount of salt (in lb) in tank T_i after time t (in min). In the time interval $[t, t+\Delta t]$:
$\Delta y_1 \approx 54 \cdot \frac{1}{2} \cdot \Delta t - 72 \cdot \frac{y_1}{24} \cdot \Delta t + 18 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_1' = 27 - 3y_1 + 2y_2$. Also, $y_1(0) = 3$.
$\Delta y_2 \approx 72 \cdot \frac{y_1}{24} \cdot \Delta t - 72 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_2' = 3y_1 - 8y_2$. Also, $y_2(0) = 0$.

One strategy to solve this system is to first combine the two DEs to get a single equation for y_1.

- From the first DE, we get $y_2 = \frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2}$.
- Using this in the second DE, we obtain $\left(\frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2}\right)' = 3y_1 - 8\left(\frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2}\right)$.
  Simplified, this is $y_1'' + 11y_1' + 18y_1 = 216$.
- We already have the initial condition $y_1(0) = 3$. We get a second one by combining $y_2 = \frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2}$ with $y_2(0) = 0$ to get $0 = y_2(0) = \frac{1}{2}y_1'(0) + \frac{3}{2}y_1(0) - \frac{27}{2} = \frac{1}{2}y_1'(0) - 9$, which simplifies to $y_1'(0) = 18$.
- The IVP $y_1'' + 11y_1' + 18y_1 = 216$ with initial conditions $y_1(0) = 3$ and $y_1'(0) = 18$ is one that we can solve!
  - The general solution of the corresponding homogeneous equation is $y_h = C_1 e^{-2t} + C_2 e^{-9t}$.
  - The simplest particular solution is of the form $y_p = C$. Plugging into the DE, we find $y_p = \frac{216}{18} = 12$.
  - Hence, the general solution to the (inhomogeneous) DE is $y(x) = 12 + C_1 e^{-2t} + C_2 e^{-9t}$.
    We then use the initial conditions $y(0) = 12 + C_1 + C_2 \overset{!}{=} 3$, $y'(0) = -2C_1 - 9C_2 \overset{!}{=} 18$ to find that for the unique solution of the IVP $C_1 = -9$, $C_2 = 0$.

  It has the unique solution $y_1(t) = 12 - 9e^{-2t}$.
- It follows that $y_2 = \frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2} = \frac{9}{2} - \frac{9}{2}e^{-2t}$.

Note. We could have found a particular solution with less calculations by observing (looking at "old" and "new" roots) that there must be a solution of the form $\boldsymbol{y}_p(t) = \boldsymbol{a}$. We can then find $\boldsymbol{a}$ by plugging into the differential equation. However, noticing that, for a constant solution, each tank has to have a constant concentration of 0.5lb/gal of salt, we find $\boldsymbol{y}_p(t) = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix}$.

Armin Straub
straub@southalabama.edu

Example 118. Write the system of equations

$$\begin{aligned} y_1' &= -3y_1 + 2y_2 + 27 \qquad & y_1(0) = 3 \\ y_2' &= 3y_1 - 8y_2 \qquad & y_2(0) = 0 \end{aligned}$$

from the previous problem in matrix-vector notation.

Solution. If we write $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then the system becomes

$$\boldsymbol{y}' = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 27 \\ 0 \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

Advanced comment. Here, we only use the matrix-vector notation as a device for writing the system of equations in a more compact form. However, it turns out that the matrix-vector notation makes certain techniques more transparent (just like writing a system of equations in the form $A\boldsymbol{x} = \boldsymbol{b}$ suggests introducing the matrix inverse to simply write $\boldsymbol{x} = A^{-1}\boldsymbol{b}$). For instance, the unique solution to a homogeneous linear system $\boldsymbol{y}' = M\boldsymbol{y}$ (where M is a matrix with constant entries) with initial condition $\boldsymbol{y}(0) = \boldsymbol{c}$ can be expressed as $\boldsymbol{y}(x) = e^{Mx}\boldsymbol{c}$, just as in the case of a single linear DE. Here, e^{Mx} is the **matrix exponential**. This will be one of the topics discussed in both Differential Equations II and Linear Algebra II.

Example 119. (extra) Three brine tanks T_1, T_2, T_3.

T_1 contains 20gal water with 10lb salt, T_2 40gal pure water, T_3 50gal water with 30lb salt.

T_1 is filled with 10gal/min water with 2lb/gal salt. 10gal/min well-mixed solution flows out of T_1 into T_2. Also, 10gal/min well-mixed solution flows out of T_2 into T_3. Finally, 10gal/min well-mixed solution is leaving T_3. How much salt is in the tanks after t minutes?

Solution. Let $y_i(t)$ denote the amount of salt (in lb) in tank T_i after time t (in min).

In the time interval $[t, t + \Delta t]$:

$\Delta y_1 \approx 10 \cdot 2 \cdot \Delta t - 10 \frac{y_1}{20} \cdot \Delta t$, so $y_1' = 20 - \frac{1}{2} y_1$. Also, $y_1(0) = 10$.

$\Delta y_2 \approx 10 \cdot \frac{y_1}{20} \cdot \Delta t - 10 \frac{y_2}{40} \cdot \Delta t$, so $y_2' = \frac{1}{2} y_1 - \frac{1}{4} y_2$. Also, $y_2(0) = 0$.

$\Delta y_3 \approx 10 \cdot \frac{y_2}{40} \cdot \Delta t - 10 \frac{y_3}{50} \cdot \Delta t$, so $y_3' = \frac{1}{4} y_2 - \frac{1}{5} y_3$. Also, $y_3(0) = 30$.

Using matrix notation and writing $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, this is $\boldsymbol{y}' = \begin{bmatrix} -1/2 & 0 & 0 \\ 1/2 & -1/4 & 0 \\ 0 & 1/4 & -1/5 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 10 \\ 0 \\ 30 \end{bmatrix}$.

We can actually solve this IVP!

[Do it! Start by finding y_1 from the first DE, then move on to $y_2 \ldots$]

Here, we content ourselves with finding a particular solution (and ignoring the initial conditions). The method of undetermined coefficients tells us that there is a solution of the form $\boldsymbol{y}_p(t) = \boldsymbol{a}$. Of course, we can find $\boldsymbol{a}$ by plugging into the differential equation. However, noticing that, for a constant solution, each tank has to have a concentration of 2lb/gal of salt, we find $\boldsymbol{y}_p = (40, 80, 100)$ without calculation.

Review. Modeling two connected fluid tanks

Higher-order linear DEs as first-order systems

Example 120. Write the (second-order) differential equation $y'' = 2y' + 5y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$ and $y_2 = y'$. Then $y'' = 2y' + 5y$ becomes $y_2' = 2y_2 + 5y_1$.

Therefore, $y'' = 2y' + 5y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = 5y_1 + 2y_2 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 5 & 2 \end{bmatrix} \boldsymbol{y}$.

Comment. This illustrates why we might care about systems of DEs, even if we work with only one function.

Example 121. Write the (third-order) differential equation $y''' = 3y'' - 2y' + 4y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' = 3y'' - 2y' + 4y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = 4y_1 - 2y_2 + 3y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -2 & 3 \end{bmatrix} \boldsymbol{y}$.

Example 122. Consider the following system of (second-order) initial value problems:

$$\begin{array}{l} y_1'' = 2y_1' - 3y_2' + 7y_2 \\ y_2'' = 4y_1' + y_2' - 5y_1 \end{array} \qquad y_1(0) = 2, \; y_1'(0) = 3, \; y_2(0) = -1, \; y_2'(0) = 1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

Armin Straub
straub@southalabama.edu

Example 123. Determine the general solution to $y_1' = 5y_1 + 4y_2 + e^{2x}$, $y_2' = 8y_1 + y_2$.

Solution. From the second equation it follows that $y_1 = \frac{1}{8}(y_2' - y_2)$. Using this in the first equation, we get $\frac{1}{8}(y_2'' - y_2') = \frac{5}{8}(y_2' - y_2) + 4y_2 + e^{2x}$. After multiplying with 8, this is $y_2'' - y_2' = 5(y_2' - y_2) + 32y_2 + 8e^{2x}$.

Simplified, this is $y_2'' - 6y_2' - 27y_2 = 8e^{2x}$, which is an inhomogeneous linear DE with constant coefficients which know how to solve:

- Since the "old" roots are $-3, 9$, while the "new" root is 2, there must a particular solution of the form $y_p = Ce^{2x}$. Plugging this y_p into the DE, we get $y_p'' - 6y_p' - 27y_p = (4 - 6 \cdot 2 - 27)Ce^{2x} = -35Ce^{2x} \overset{!}{=} 8e^{2x}$. Hence, $C = -\frac{8}{35}$.
- We therefore obtain $y_2 = -\frac{8}{35}e^{2x} + C_1e^{-3x} + C_2e^{9x}$ as the general solution to the inhomogeneous DE.

We can then determine y_1 as

$$\begin{aligned} y_1 &= \frac{1}{8}(y_2' - y_2) \\ &= \frac{1}{8}\left(-\frac{16}{35}e^{2x} - 3C_1e^{-3x} + 9C_2e^{9x} + \frac{8}{35}e^{2x} - C_1e^{-3x} - C_2e^{9x}\right) \\ &= -\frac{1}{35}e^{2x} - \frac{1}{2}C_1e^{-3x} + C_2e^{9x}. \end{aligned}$$

Solution. (alternative) We can also start with $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1 - \frac{1}{4}e^{2x}$ (from the first equation), although the algebra will require a little more work. In that case, we have $y_2' = \frac{1}{4}y_1'' - \frac{5}{4}y_1' - \frac{1}{2}e^{2x}$. Using this in the second equation, we get $\frac{1}{4}y_1'' - \frac{5}{4}y_1' - \frac{1}{2}e^{2x} = 8y_1 + \frac{1}{4}y_1' - \frac{5}{4}y_1 - \frac{1}{4}e^{2x}$.

Simplified, this is $y_1'' - 6y_1' - 27y_1 = e^{2x}$, which is an inhomogeneous linear DE with constant coefficients which know how to solve:

- Since the "old" roots are $-3, 9$, while the "new" root is 2, there must a particular solution of the form $y_p = Ce^{2x}$. Plugging this y_p into the DE, we get $y_p'' - 6y_p' - 27y_p = (4 - 6 \cdot 2 - 27)Ce^{2x} = -35Ce^{2x} \overset{!}{=} e^{2x}$. Hence, $C = -\frac{1}{35}$.
- We therefore obtain $y_1 = -\frac{1}{35}e^{2x} + C_1e^{-3x} + C_2e^{9x}$ as the general solution to the inhomogeneous DE.

We can then determine y_2 as

$$\begin{aligned} y_2 &= \frac{1}{4}y_1' - \frac{5}{4}y_1 - \frac{1}{4}e^{2x} \\ &= \frac{1}{4}\left(-\frac{2}{35}e^{2x} - 3C_1e^{-3x} + 9C_2e^{9x}\right) - \frac{5}{4}\left(-\frac{1}{35}e^{2x} + C_1e^{-3x} + C_2e^{9x}\right) - \frac{1}{4}e^{2x} \\ &= -\frac{8}{35}e^{2x} - 2C_1e^{-3x} + C_2e^{9x}. \end{aligned}$$

Important. Make sure you can explain why both of our solutions are equivalent!

Armin Straub
straub@southalabama.edu

Example 124.

(a) Determine the general solution to $y_1' = 5y_1 + 4y_2$, $y_2' = 8y_1 + y_2$.

Comment. In matrix form, with $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, this is $\boldsymbol{y}' = \begin{bmatrix} 5 & 4 \\ 8 & 1 \end{bmatrix} \boldsymbol{y}$.

(b) Solve the IVP $y_1' = 5y_1 + 4y_2$, $y_2' = 8y_1 + y_2$, $y_1(0) = 0$, $y_2(0) = 1$.

Solution.

(a) Note that this is the homogeneous system corresponding to the previous problem. It therefore follows from our previous solution (the latter one) that $y_1 = C_1 e^{-3x} + C_2 e^{9x}$ and $y_2 = -2C_1 e^{-3x} + C_2 e^{9x}$ is the general solution of the homogeneous system.

(b) We already have the general solutions y_1, y_2 to the two DEs. We need to determine the (unique) values of C_1 and C_2 to match the initial conditions: $y_1(0) = C_1 + C_2 \overset{!}{=} 0$, $y_2(0) = -2C_1 + C_2 \overset{!}{=} 1$

We solve these two equations and find $C_1 = -\frac{1}{3}$ and $C_2 = \frac{1}{3}$.

The unique solution to the IVP therefore is $y_1 = -\frac{1}{3} e^{-3x} + \frac{1}{3} e^{9x}$ and $y_2 = \frac{2}{3} e^{-3x} + \frac{1}{3} e^{9x}$.

Armin Straub
straub@southalabama.edu

Notes for Lecture 31

Fri, 11/10/2023

Excursion: Euler's identity

Let's revisit Euler's identity from Theorem 79.

Theorem 125. (Euler's identity) $e^{ix} = \cos(x) + i\sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] □

On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Example 126. Where do trig identities like $\sin(2x) = 2\cos(x)\sin(x)$ or $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ (and infinitely many others!) come from?

Short answer: they all come from the simple exponential law $e^{x+y} = e^x e^y$.

Let us illustrate this in the simple case $(e^x)^2 = e^{2x}$. Observe that

$$\begin{aligned} e^{2ix} &= \cos(2x) + i\sin(2x) \\ e^{ix}e^{ix} &= [\cos(x) + i\sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i\cos(x)\sin(x). \end{aligned}$$

Comparing imaginary parts (the "stuff with an i"), we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.
Likewise, comparing real parts, we read off $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(Use $\cos^2(x) + \sin^2(x) = 1$ to derive $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ from the last equation.)

Challenge. Can you find a triple-angle trig identity for $\cos(3x)$ and $\sin(3x)$ using $(e^x)^3 = e^{3x}$?
Or, use $e^{i(x+y)} = e^{ix}e^{iy}$ to derive $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x + y) = \ldots$

Realize that the complex number $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ corresponds to the point $(\cos(\theta), \sin(\theta))$.
These are precisely the points on the unit circle!

Recall that a point (x, y) can be represented using **polar coordinates** (r, θ), where r is the distance to the origin and θ is the angle with the x-axis.

Then, $x = r\cos\theta$ and $y = r\sin\theta$.

Every complex number z can be written in **polar form** as $z = re^{i\theta}$, with $r = |z|$.

Why? By comparing with the usual polar coordinates ($x = r\cos\theta$ and $y = r\sin\theta$), we can write

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}.$$

In the final step, we used Euler's identity.

Armin Straub
straub@southalabama.edu

The Laplace transform

Definition 127. The **Laplace transform** of a function $f(t)$, $t \geqslant 0$, is defined as the new function

$$F(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t.$$

We also write $\mathcal{L}(f(t)) = F(s)$.

Note that, in order for the integral to exist, $f(t)$ should be, say, piecewise continuous and of at most exponential growth. That's true for most of the functions we are interested in (and so we will not dwell on this issue).

$f(t)$	$F(s)$
e^{at}	$\frac{1}{s-a}$
1	$\frac{1}{s}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
$f'(t)$	$s F(s) - f(0)$
$f''(t)$	$s^2 F(s) - s f(0) - f'(0)$

First entries in the Laplace transform table

In this section, we will discuss and obtain the entries in the table of the most basic Laplace transforms that we compiled after Definition 127.

Example 128. Show that $\mathcal{L}(e^{at}) = \frac{1}{s-a}$.

In particular, in the special case $a = 0$, we have $\mathcal{L}(1) = \frac{1}{s}$.

Solution. $\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at} \mathrm{d}t = \int_0^\infty e^{(a-s)t} \mathrm{d}t = \left[\frac{1}{a-s} e^{(a-s)t}\right]_{t=0}^\infty = 0 - \frac{1}{a-s} = \frac{1}{s-a}$

Comment. Note that we needed $a - s < 0$ in order for the integral to converge. Hence the Laplace transform has domain $s > a$. (During this introduction, we will not care too much about these technical details.)

In particular. Note that setting $a = 0$ shows that $\mathcal{L}(1) = \frac{1}{s}$.

Example 129. (linearity) Show that $\mathcal{L}(c_1 f_1(t) + c_2 f_2(t)) = c_1 F_1(s) + c_2 F_2(s)$.

This means that the Laplace transform is a **linear operator** (like the derivative or the integral).

Solution.

$$\begin{aligned}\mathcal{L}(c_1 f_1(t) + c_2 f_2(t)) &= \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t)) \mathrm{d}t \\ &= c_1 \underbrace{\int_0^\infty e^{-st} f_1(t) \mathrm{d}t}_{F_1(s)} + c_2 \underbrace{\int_0^\infty e^{-st} f_2(t) \mathrm{d}t}_{F_2(x)}\end{aligned}$$

Example 130. (extra) Show that $\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$ and $\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$.

Solution. By Euler's identity, $e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$. Hence, by linearity,

$$\mathcal{L}(e^{i\omega t}) = \mathcal{L}(\cos(\omega t)) + i \mathcal{L}(\sin(\omega t)).$$

On the other hand,

$$\mathcal{L}(e^{i\omega t}) = \frac{1}{s - i\omega} = \frac{s + i\omega}{s^2 + \omega^2} = \frac{s}{s^2+\omega^2} + i \frac{\omega}{s^2+\omega^2}.$$

Matching real and imaginary parts, we find $\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$ and $\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$.

Armin Straub
straub@southalabama.edu

Example 131. Determine $\mathcal{L}(e^{3t} - 7e^{-2t})$.

Solution. $\mathcal{L}(e^{3t} - 7e^{-2t}) = \mathcal{L}(e^{3t}) - 7\mathcal{L}(e^{-2t}) = \frac{1}{s-3} - \frac{7}{s+2}$

Comment. Note that, once we write $\frac{1}{s-3} - \frac{7}{s+2} = -\frac{6s-23}{s^2-s-6}$ it is no longer visibly clear which function we have taken the Laplace transform of. We discuss reversing this process in the next section.

Example 132. (extra) Determine $\mathcal{L}(3\cos(2t) - 5\sin(2t))$.

Solution. $\mathcal{L}(3\cos(2t) - 5\sin(2t)) = 3\mathcal{L}(\cos(2t)) - 5\mathcal{L}(\sin(2t)) = 3\frac{s}{s^2+4} - 5\frac{2}{s^2+4} = \frac{3s-10}{s^2+4}$

Example 133. Show that $\mathcal{L}(f'(t)) = sF(s) - f(0)$.

Solution. Using integration by parts,

$$\mathcal{L}(f'(t)) = \int_0^\infty e^{-st} f'(t)\mathrm{d}t = \left[e^{-st} f(t)\right]_{t=0}^\infty + \int_0^\infty s e^{-st} f(t)\mathrm{d}t = sF(s) - f(0).$$

Higher derivatives. In order to obtain the Laplace transform of higher derivatives, we can iterate. For instance,

$$\mathcal{L}(f''(t)) = s\mathcal{L}(f'(t)) - f'(0) = s[sF(s) - f(0)] - f'(0) = s^2F(s) - sf(0) - f'(0).$$

Armin Straub
straub@southalabama.edu

The inverse Laplace transform

Theorem 134. (uniqueness of Laplace transforms) If $\mathcal{L}(f_1(t)) = \mathcal{L}(f_2(t))$, then $f_1(t) = f_2(t)$.
Hence, we can recover $f(t)$ from $F(s)$. We write $\mathcal{L}^{-1}(F(s)) = f(t)$.

We say that $f(t)$ is the **inverse Laplace transform** of $F(s)$.

Advanced comment. This uniqueness is true for continuous functions f_1, f_2. It is also true for piecewise continuous functions except at those values of t for which there is a discontinuity. (Note that redefining $f(t)$ at a single point, will not change its Laplace transform.)

Example 135. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{5}{s+3}\right)$.

Solution. In other words, if $F(s) = \frac{5}{s+3}$, what is $f(t)$?
$\mathcal{L}^{-1}\left(\frac{5}{s+3}\right) = 5\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = 5e^{-3t}$

Example 136. (extra) Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{3s-7}{s^2+4}\right)$.

Solution. In other words, if $F(s) = \frac{3s-7}{s^2+4}$, what is $f(t)$?
$F(s) = 3\frac{s}{s^2+2^2} - \frac{7}{2}\frac{2}{s^2+2^2}$. Hence, $f(t) = 3\cos(2t) - \frac{7}{2}\sin(2t)$.

Example 137. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right)$.

Solution. Note that $s^2 - s - 6 = (s-3)(s+2)$. We use **partial fractions** to write $-\frac{6s-23}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$.
We find the coefficients (see brief review below) as

$$A = -\frac{6s-23}{s+2}\bigg|_{s=3} = 1, \quad B = -\frac{6s-23}{s-3}\bigg|_{s=-2} = -7.$$

Hence $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-3} - \frac{7}{s+2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - 7\mathcal{L}^{-1}\left(\frac{7}{s+2}\right) = e^{3t} - 7e^{-2t}$.

Review. In order to find A, we multiply $-\frac{6s-23}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$ by $s-3$ to get $-\frac{6s-23}{s+2} = A + \frac{B(s-3)}{s+2}$.
We then set $s=3$ to find A as above.

Comment. Compare with Example 131 where we considered the same functions.

Example 138. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{s+13}{s^2-s-2}\right)$.

Solution. Note that $s^2 - s - 2 = (s-2)(s+1)$. We use partial fractions to write $\frac{s+13}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$.
We find the coefficients as

$$A = \frac{s+13}{s+1}\bigg|_{s=2} = 5, \quad B = \frac{s+13}{s-2}\bigg|_{s=-1} = -4.$$

Hence $\mathcal{L}^{-1}\left(\frac{s+13}{s^2-s-2}\right) = \mathcal{L}^{-1}\left(\frac{5}{s+1} - \frac{4}{s-2}\right) = 5e^t - 4e^{2t}$.

Armin Straub
straub@southalabama.edu

Solving simple DEs using the Laplace transform

In the following examples, we write $Y(s)$ for the Laplace transform of $y(t)$.

Recall from our Laplace transform table that this implies that the Laplace transform of $y'(t)$ is $sY(s) - y(0)$.

Example 139. Solve the (very simple) IVP $y'(t) - 2y(t) = 0$, $y(0) = 7$.

At this point, you might be able to "see" right away that the unique solution is $y(t) = 7e^{2t}$.

Solution. (old style) The characteristic root is 2, so that the general solution is $y(t) = Ce^{2t}$. Using the initial condition, we find that $C = 7$, so that $y(t) = 7e^{2t}$.

Solution. (Laplace style) $y' - 2y = 0$ transforms into

$$\mathcal{L}(y'(t) - 2y(t)) = \mathcal{L}(y'(t)) - 2\mathcal{L}(y(t)) = sY(s) - y(0) - 2Y(s) = (s-2)Y(s) - 7 = 0.$$

This is an algebraic equation for $Y(s)$. It follows that $Y(s) = \frac{7}{s-2}$. By inverting the Laplace transform, we conclude that $y(t) = 7e^{2t}$.

Armin Straub
straub@southalabama.edu

Example 140. (review) Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right)$.

Solution. Note that $s^2-s-6=(s-3)(s+2)$. We use **partial fractions** to write $-\frac{6s-23}{(s-3)(s+2)}=\frac{A}{s-3}+\frac{B}{s+2}$. We find the coefficients (see brief review below) as

$$A=-\frac{6s-23}{s+2}\bigg|_{s=3}=1,\quad B=-\frac{6s-23}{s-3}\bigg|_{s=-2}=-7.$$

Hence $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right)=\mathcal{L}^{-1}\left(\frac{1}{s-3}-\frac{7}{s+2}\right)=\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)-7\mathcal{L}^{-1}\left(\frac{7}{s+2}\right)=e^{3t}-7e^{-2t}$.

Review. In order to find A, we multiply $-\frac{6s-23}{(s-3)(s+2)}=\frac{A}{s-3}+\frac{B}{s+2}$ by $s-3$ to get $-\frac{6s-23}{s+2}=A+\frac{B(s-3)}{s+2}$. We then set $s=3$ to find A as above.

Comment. Compare with Example 131 where we considered the same functions.

Example 141. Solve the IVP $y''-3y'+2y=e^{-t}$, $y(0)=0$, $y'(0)=1$.

Solution. (old style) The characteristic polynomial $D^2-3D+2=(D-1)(D-2)$ has ("old") roots $1,2$. The "new" root is -1. Since there is no duplication, there must be a particular solution of the form $y_p(t)=Ae^{-t}$.

To determine A, we plug into the DE $y_p''-3y_p'+2y_p=6Ae^{-t}\overset{!}{=}e^{-t}$ and conclude $A=\frac{1}{6}$.

The general solution thus is $y(t)=\frac{1}{6}e^{-t}+C_1e^t+C_2e^{2t}$. We need to find C_1 and C_2 using the initial conditions.

Solving $y(0)=\frac{1}{6}+C_1+C_2\overset{!}{=}0$ and $y'(0)=-\frac{1}{6}+C_1+2C_2\overset{!}{=}1$, we find $C_2=\frac{4}{3}$ and $C_1=-\frac{3}{2}$.

Hence, the unique solution to the IVP is $y(t)=\frac{1}{6}e^{-t}-\frac{3}{2}e^t+\frac{4}{3}e^{2t}$.

Solution. (Laplace style) The differential equation (plus initial conditions!) transforms as follows:

$$\begin{aligned}
\mathcal{L}(y''(t))-3\,\mathcal{L}(y'(t))+2\mathcal{L}(y(t)) &= \mathcal{L}(e^{-t})\\
s^2Y(s)-sy(0)-y'(0)-3(\,sY(s)-y(0)\,)+2Y(s) &= \frac{1}{s+1}\\
(s^2-3s+2)Y(s) &= 1+\frac{1}{s+1}=\frac{s+2}{s+1}\\
Y(s) &= \frac{s+2}{(s^2-3s+2)(s+1)}\\
&= \frac{s+2}{(s-1)(s-2)(s+1)}
\end{aligned}$$

To find $y(t)$, we use partial fractions to write $Y(s)=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+1}$. We find the coefficients as

$$A=\frac{s+2}{(s-2)(s+1)}\bigg|_{s=1}=-\frac{3}{2},\quad B=\frac{s+2}{(s-1)(s+1)}\bigg|_{s=2}=\frac{4}{3},\quad C=\frac{s+2}{(s-1)(s-2)}\bigg|_{s=-1}=\frac{1}{6}.$$

Hence, $y(t)=\mathcal{L}^{-1}\left(\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+1}\right)=Ae^t+Be^{2t}+Ce^{-t}=\frac{1}{6}e^{-t}-\frac{3}{2}e^t+\frac{4}{3}e^{2t}$, as above.

Comment. Note the factor s^2-3s+2 in front of $Y(s)$ when we transformed the DE. This is the characteristic polynomial. Can you see how the "old" and "new" roots show up in the Laplace transform approach?

Example 142. Consider the IVP $y''-3y'+2y=e^{-t}$, $y(0)=0$, $y'(0)=1$.
Determine the Laplace transform of the unique solution.

Solution. We just did that! By transforming the DE, we found that $Y(s)=\frac{s+2}{(s-1)(s-2)(s+1)}$.

Example 143. Consider the IVP $y'' - 3y' + y = 2e^{5t}$, $y(0) = -1$, $y'(0) = 4$.

Determine the Laplace transform of the unique solution.

Solution. The DE $y'' - 3y' + y = 2e^{5t}$ (plus initial conditions!) transforms into

$$s^2Y - sy(0) - y'(0) - 3(sY - y(0)) + Y = (s^2 - 3s + 1)Y + (s - 7) = \frac{2}{s-5}.$$

Accordingly, $Y(s) = \frac{1}{s^2 - 3s + 1}\left[\frac{2}{s-5} - s + 7\right]$ is the Laplace transform of the unique solution to the IVP.

Comment. The characteristic roots are $(3 \pm \sqrt{5})/2$. As a result, the solution $y(t)$ will be rather unpleasant to write down by hand, with coefficients that are not rational numbers. By contrast, the above Laplace transform can be expressed without irrational numbers.

Comment. Depending on what we intend to do with the solution, we might not even need $y(t)$ but might instead be able to extract what we want from its Laplace transform $Y(s)$.

Handling discontinuities with the Laplace transform

Let $u_a(t) = \begin{cases} 1, & \text{if } t \geqslant a, \\ 0, & \text{if } t < a, \end{cases}$ be the **unit step function**. Throughout, we assume that $a \geqslant 0$.

Comment. The special case $u_0(t)$ is also known as the **Heaviside function**, after Oliver Heaviside who, among many other things, coined terms like conductance and impedance. Note that $u_a(t) = u_0(t-a)$.

Example 144. If $a < b$, then $u_a(t) - u_b(t) = \begin{cases} 1, & \text{if } a \leqslant t < b, \\ 0, & \text{otherwise.} \end{cases}$

Comment. See Example 147 for how to write piecewise-defined functions as combinations of unit step functions.

The following is a useful addition to our table of Laplace transforms:

Example 145. (new entry) We add the following to our table of Laplace transforms:

$$\begin{aligned} \mathcal{L}(u_a(t) f(t-a)) &= \int_a^\infty e^{-st} f(t-a) \mathrm{d}t = \int_0^\infty e^{-s(\tilde{t}+a)} f(\tilde{t}) \mathrm{d}\tilde{t} \\ &= e^{-as} \int_0^\infty e^{-s\tilde{t}} f(\tilde{t}) \mathrm{d}\tilde{t} = e^{-as} F(s) \end{aligned}$$

Comment. Note that the graph of $u_a(t) f(t-a)$ is the same as $f(t)$ but delayed by a (make a sketch!).

In particular. $\mathcal{L}(u_a(t)) = \frac{e^{-sa}}{s}$

Thus equipped, we can solve differential equations featuring certain kinds of discontinuities.

Note that the DE in our next example describes the motion of a mass on a spring with damping, where the external force is zero except for the time interval $[2, 3)$ when we suddenly have a force equal to 5.

Example 146. Determine the Laplace transform of the unique solution to the initial value problem

$$y'' + 5y' + 6y = \begin{cases} 5, & \text{if } 2 \leqslant t < 3, \\ 0, & \text{otherwise,} \end{cases} \qquad y(0) = -4, \quad y'(0) = 8.$$

Solution. First, we observe that the right-hand side of the differential equation can be written as $5(u_2(t) - u_3(t))$. It follows from the Laplace transform table that $\mathcal{L}(u_a(t)) = e^{-as} \frac{1}{s}$ (using the entry for $u_a(t) f(t-a)$ with $f(t) = 1$). Consequently, $\mathcal{L}(5(u_2(t) - u_3(t))) = 5e^{-2s} \frac{1}{s} - 5e^{-3s} \frac{1}{s} = \frac{5}{s}(e^{-2s} - e^{-3s})$.

Taking the Laplace transform of both sides of the DE, we therefore get

$$s^2 Y(s) - s y(0) - y'(0) + 5(sY(s) - y(0)) + 6Y(s) = \frac{5}{s}(e^{-2s} - e^{-3s}),$$

which using the initial values simplifies to

$$(s^2 + 5s + 6) Y(s) + 4s - 8 + 5 \cdot 4 = \frac{5}{s}(e^{-2s} - e^{-3s}).$$

We conclude that the Laplace transform of the unique solution is

$$Y(s) = \frac{1}{s^2 + 5s + 6} \left[\frac{5}{s}(e^{-2s} - e^{-3s}) - 4s - 12 \right].$$

First challenge. Take the inverse Laplace transform to find $y(t)$! (See Examples 148 and 149.)

Second challenge. Solve the DE without using Laplace transforms! (First, solve the IVP for $t < 2$ in which case we have no external force. That tells us what $y(2)$ and $y'(2)$ should be. Using these as the new initial conditions, solve the IVP for $t \in [2, 3)$. Then, using $y(3)$ and $y'(3)$, solve the IVP for $t \geqslant 3$. In the end, you will have found the solution $y(t)$ in three pieces. On the other hand, the Laplace transform allows us to avoid working piece-by-piece.)

Armin Straub
straub@southalabama.edu

The next example illustrates that any piecewise defined function can be written using a single formula involving step functions. This is based on the simple observation from Example 144 that $u_a(t) - u_b(t)$ is a function which is 1 on the interval $[a, b)$ but zero everywhere else.

Comment. For our present purposes, we don't really care about the precise value of a function at a single point. Specifically, it doesn't really matter which value the function $u_a(t) - u_b(t)$ takes at $t = b$ (technically, the value is 0 but it may as well be 1 since there is a discontinuity at $t = b$).

Example 147. Write $f(t) = \begin{cases} 0, & \text{if } t < 0, \\ t^2, & \text{if } 0 \leqslant t < 1, \\ 3, & \text{if } 1 \leqslant t < 2, \\ \cos(t-2), & \text{if } t \geqslant 2, \end{cases}$ as a combination of unit step functions.

Solution. $f(t) = t^2(u_0(t) - u_1(t)) + 3(u_1(t) - u_2(t)) + \cos(t-2)u_2(t)$

Homework. Compute the Laplace transform of $f(t)$ from here. Note that, for instance, to find $\mathcal{L}(t^2 u_1(t))$, we want to use $\mathcal{L}(u_a(t) f(t-a)) = e^{-sa} F(s)$ with $a = 1$ and $f(t-1) = t^2$. Then, $f(t) = (t+1)^2 = t^2 + 2t + 1$ has Laplace transform $F(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$. Combined, we get $\mathcal{L}(t^2 u_1(t)) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$.

Example 148. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+3}\right)$.

Solution. $\frac{1}{s+3}$ is the Laplace transform of e^{-3t}. Hence, $\frac{e^{-2s}}{s+3}$ is the Laplace transform of e^{-3t} delayed by 2. In other words, $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+3}\right) = u_2(t) e^{-3(t-2)}$.

Comment. Note that this is one of the terms in our solution $Y(s)$ in Example 146 (because $s^2 + 5s + 6 = (s+2)(s+3)$). Can you determine the full inverse Laplace transform of $Y(s)$?

Example 149. Solve the IVP $y'' + 3y' + 2y = f(t)$, $y(0) = y'(0) = 0$ with $f(t) = \begin{cases} 1, & 3 \leqslant t < 4, \\ 0, & \text{otherwise.} \end{cases}$

Solution. First, we write $f(t) = u_3(t) - u_4(t)$. We can now take the Laplace transform of the DE to get

$$s^2 Y(s) - s y(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) = \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s} = (e^{-3s} - e^{-4s})\frac{1}{s}.$$

Using that $s^2 + 3s + 2 = (s+1)(s+2)$, we find

$$Y(s) = (e^{-3s} - e^{-4s})\frac{1}{s(s+1)(s+2)} = (e^{-3s} - e^{-4s})\left[\frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}\right],$$

where A, B, C are determined by partial fractions (we compute the values below). Taking the inverse Laplace transform of each of the six terms in this product, as in Example 148, we find

$$y(t) = A(u_3(t) - u_4(t)) + B(u_3(t)e^{-(t-3)} - u_4(t)e^{-(t-4)}) + C(u_3(t)e^{-2(t-3)} - u_4(t)e^{-2(t-4)}).$$

If preferred, we can express this as $y(t) = \begin{cases} 0, & \text{if } t < 3, \\ A + Be^{-(t-3)} + Ce^{-2(t-3)}, & \text{if } 3 \leqslant t < 4, \\ B(e^{-(t-3)} - e^{-(t-4)}) + C(e^{-2(t-3)} - e^{-2(t-4)}) & \text{if } t \geqslant 4. \end{cases}$

Finally, $A = \left.\frac{1}{(s+1)(s+2)}\right|_{s=0} = \frac{1}{2}$, $B = \left.\frac{1}{s(s+2)}\right|_{s=-1} = -1$, $C = \left.\frac{1}{s(s+1)}\right|_{s=-2} = \frac{1}{2}$.

Comment. Check that these values make $y(t)$ a continuous function (as it should be for physical reasons).

Example 150. Determine the Laplace transform $\mathcal{L}(e^{rt} u_a(t))$.

Solution. Write $e^{rt} u_a(t) = f(t-a) u_a(t)$ with $f(t) = e^{r(t+a)} = e^{ra} e^{rt}$. Since $F(s) = \mathcal{L}(f(t)) = \frac{e^{ra}}{s-r}$, we have

$$\mathcal{L}(e^{rt} u_a(t)) = e^{-sa} F(s) = \frac{e^{-(s-r)a}}{s-r}.$$

Example 151. (extra practice) Determine the Laplace transform of the unique solution to the initial value problem

$$y'' - 6y' + 5y = \begin{cases} 3e^{-2t}, & \text{if } 1 \leqslant t < 4, \\ 0, & \text{otherwise,} \end{cases} \qquad y(0) = 2, \quad y'(0) = -1.$$

Solution. First, we write the right-hand side of the differential equation as $f(t) := 3e^{-2t}(u_1(t) - u_4(t))$. By Example 150, the Laplace transform of $f(t)$ is $\mathcal{L}(f(t)) = 3\frac{e^{-(s+2)}}{s+2} - 3\frac{e^{-4(s+2)}}{s+2} = \frac{3}{s+2}(e^{-(s+2)} - e^{-4(s+2)})$.
Taking the Laplace transform of both sides of the DE, we therefore get

$$s^2 Y(s) - sy(0) - y'(0) - 6(sY(s) - y(0)) + 5Y(s) = \frac{3}{s+2}(e^{-(s+2)} - e^{-4(s+2)}),$$

which using the initial values simplifies to

$$(s^2 - 6s + 5)Y(s) - 2s + 13 = \frac{3}{s+2}(e^{-(s+2)} - e^{-4(s+2)}).$$

We conclude that the Laplace transform of the unique solution is

$$Y(s) = \frac{1}{s^2 - 6s + 5}\left[\frac{3}{s+2}(e^{-(s+2)} - e^{-4(s+2)}) + 2s - 13\right].$$

Further entries in the Laplace transform table

Finally, we expand our table of Laplace transforms to the following:

$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
e^{at}	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{at}f(t)$	$F(s-a)$
$t f(t)$	$-F'(s)$
$u_a(t)f(t-a)$	$e^{-as}F(s)$

Example 152. (new entry) We add the following to our table of Laplace transforms:

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)\mathrm{d}t = \int_0^\infty e^{-(s-a)t}f(t)\mathrm{d}t = F(s-a)$$

Example 153. (new entry) We also add the following to our table of Laplace transforms:

$$\mathcal{L}(t f(t)) = \int_0^\infty e^{-st}t f(t)\mathrm{d}t = \int_0^\infty -\frac{\mathrm{d}}{\mathrm{d}s}e^{-st}f(t)\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st}f(t)\mathrm{d}t = -F'(s)$$

In particular,

$$\begin{aligned}\mathcal{L}(t) &= \mathcal{L}(t\cdot 1) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s} = \frac{1}{s^2}\\ \mathcal{L}(t^2) &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s^2} = \frac{2}{s^3}\\ &\vdots\\ \mathcal{L}(t^n) &= \frac{n!}{s^{n+1}}.\end{aligned}$$

Example 154. Determine the Laplace transform $\mathcal{L}((t-3)e^{2t})$.

Solution. $\mathcal{L}((t-3)e^{2t}) = \mathcal{L}(te^{2t}) - 3\mathcal{L}(e^{2t}) = \frac{1}{(s-2)^2} - \frac{3}{s-2}$

Here, we combined $\mathcal{L}(t f(t)) = -F'(s)$ with $\mathcal{L}(e^{2t}) = \frac{1}{s-2}$ to get $\mathcal{L}(te^{2t}) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s-2} = \frac{1}{(s-2)^2}$.

Alternative. Combine $\mathcal{L}(t-3) = \frac{1}{s^2} - \frac{3}{s}$ and $\mathcal{L}(f(t)e^{2t}) = F(s-2)$ to again get $\mathcal{L}((t-3)e^{2t}) = \frac{1}{(s-2)^2} - \frac{3}{s-2}$.

Example 155. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right)$.

Solution. $\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right) = e^{3t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = te^{3t}$.

Example 156. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{(s-3)^2}\right)$.

Solution. It follows from the previous example that $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{(s-3)^2}\right) = u_2(t)(t-2)\,e^{3(t-2)}$.

Armin Straub
straub@southalabama.edu

Example 157. (bonus) Solve the IVP $y'' - 3y' + 2y = e^t$, $y(0) = 0$, $y'(0) = 1$.

Solution. (old style, outline) The characteristic polynomial $D^2 - 3D + 2 = (D - 1)(D - 2)$. Since there is duplication, we have to look for a particular solution of the form $y_p = Ate^t$. To determine A, we need to plug into the DE (we find $A = -1$). Then, the general solution is $y(t) = Ate^t + C_1 e^t + C_2 e^{2t}$, and the initial conditions determine C_1 and C_2 (we find $C_1 = -2$ and $C_2 = 2$).

Solution. (Laplace style)

$$\begin{aligned}
\mathcal{L}(y''(t)) - 3\mathcal{L}(y'(t)) + 2\mathcal{L}(y(t)) &= \mathcal{L}(e^t)\\
s^2 Y(s) - s y(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) &= \frac{1}{s-1}\\
(s^2 - 3s + 2)Y(s) &= 1 + \frac{1}{s-1} = \frac{s}{s-1}\\
Y(s) &= \frac{s}{(s-1)^2(s-2)}
\end{aligned}$$

To find $y(t)$, we again use partial fractions. We find $Y(s) = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}$ with coefficients (why?!)

$$C = \frac{s}{(s-1)^2}\bigg|_{s=2} = 2, \quad A = \frac{s}{s-2}\bigg|_{s=1} = -1, \quad B = \frac{\mathrm{d}}{\mathrm{d}s}\frac{s}{s-2}\bigg|_{s=1} = \frac{-2}{(s-2)^2}\bigg|_{s=1} = -2.$$

Finally, $y(t) = \mathcal{L}^{-1}\left(\frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}\right) = Ate^t + Be^t + Ce^{2t} = -(t+2)e^t + 2e^{2t}$.

More details on the partial fractions with a repeated root. Above we computed A, B, C so that

$$\frac{s}{(s-1)^2(s-2)} = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}.$$

- We can compute C as before by multiplying both sides with $s - 2$ and then setting $s = 2$.
- Similarly, we can compute A by multiplying both sides with $(s-1)^2$ and then setting $s = 1$.
- To compute B, multiply both sides by $(s-1)^2$ (as for A) to get

$$\frac{s}{(s-2)} = A + B(s-1) + \frac{C(s-1)^2}{s-2}.$$

Now, we take the derivative on both sides (so that A goes away) to get

$$\frac{-2}{(s-2)^2} = B + \frac{C(2(s-1)(s-2) - (s-1)^2)}{(s-2)^2}$$

and we find B by setting $s = 1$.

Comment. In fact, the term involving C had to drop out when plugging in $s = 1$, even after taking a derivative. That's because, after multiplying with $(s-1)^2$, that term has a double root at $s = 1$. When taking a derivative, it therefore still has a (single) root at $s = 1$.

Armin Straub
straub@southalabama.edu

We solved the following system in Example 123 using elimination and our method for solving linear DEs with constant coefficients based on characteristic roots.

Example 158. (extra) Solve the system $y_1' = 5y_1 + 4y_2$, $y_2' = 8y_1 + y_2$, $y_1(0) = 0$, $y_2(0) = 1$.

Solution. (using Laplace transforms) $y_1' = 5y_1 + 4y_2$ transforms into $sY_1 - \underbrace{y_1(0)}_{=0} = 5Y_1 + 4Y_2$.

Likewise, $y_2' = 8y_1 + y_2$ transforms into $sY_2 - \underbrace{y_2(0)}_{=1} = 8Y_1 + Y_2$.

The transformed equations are regular equations that we can solve for Y_1 and Y_2.

For instance, by the first equation, $Y_2 = \frac{1}{4}(s-5)Y_1$.

Used in the second equation, we get $\underbrace{-8Y_1 + \frac{1}{4}(s-1)(s-5)Y_1}_{=\frac{1}{4}(s^2-6s-27)=\frac{1}{4}(s+3)(s-9)} = 1$ so that $Y_1 = \frac{4}{(s+3)(s-9)}$.

Hence, the system is solved by $Y_1 = \frac{4}{(s+3)(s-9)}$ and $Y_2 = \frac{1}{4}(s-5)Y_1 = \frac{s-5}{(s+3)(s-9)}$.

As a final step, we need to take the inverse Laplace transform to get $y_1(t) = \mathcal{L}^{-1}(Y_1(s))$ and $y_2(t) = \mathcal{L}^{-1}(Y_2(s))$.

Using partial fractions, $Y_1(s) = \frac{4}{(s+3)(s-9)} = -\frac{1}{3} \cdot \frac{1}{s+3} + \frac{1}{3} \cdot \frac{1}{s-9}$ so that $y_1(t) = -\frac{1}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Similarly, $Y_2(s) = \frac{s-5}{(s+3)(s-9)} = \frac{2}{3} \cdot \frac{1}{s+3} + \frac{1}{3} \cdot \frac{1}{s-9}$ so that $y_2(t) = \frac{2}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Solution. (old solution, for comparison) Since $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1$ (from the first eq.), we have $y_2' = \frac{1}{4}y_1'' - \frac{5}{4}y_1'$.

Using these in the second equation, we get $\frac{1}{4}y_1'' - \frac{5}{4}y_1' = 8y_1 + \frac{1}{4}y_1' - \frac{5}{4}y_1$.
Simplified, this is $y_1'' - 6y_1' - 27y_1 = 0$.
This is a homogeneous linear DE with constant coefficients. The characteristic roots are $-3, 9$.
We therefore obtain $y_1 = C_1e^{-3t} + C_2e^{9t}$ as the general solution.
Thus, $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1 = \frac{1}{4}(-3C_1e^{-3t} + 9C_2e^{9t}) - \frac{5}{4}(C_1e^{-3t} + C_2e^{9t}) = -2C_1e^{-3t} + C_2e^{9t}$.
We determine the (unique) values of C_1 and C_2 using the initial conditions:

$y_1(0) = C_1 + C_2 \stackrel{!}{=} 0$
$y_2(0) = -2C_1 + C_2 \stackrel{!}{=} 1$
We solve these two equations and find $C_1 = -\frac{1}{3}$ and $C_2 = \frac{1}{3}$.

The unique solution to the IVP therefore is $y_1(t) = -\frac{1}{3}e^{-3t} + \frac{1}{3}e^{9t}$ and $y_2(t) = \frac{2}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Armin Straub
straub@southalabama.edu

Hyperbolic sine and cosine

Review. Euler's formula states that $e^{it} = \cos(t) + i\sin(t)$.

Recall that a function $f(t)$ is **even** if $f(-t) = f(t)$. Likewise, it is **odd** if $f(-t) = -t$.

Note that $f(t) = t^n$ is even if and only if n is even. Likewise, $f(t) = t^n$ is odd if and only if n is odd. That's where the names are coming from.

Any function $f(t)$ can be decomposed into an even and an odd part as follows:

$$f(t) = f_{\text{even}}(t) + f_{\text{odd}}(t), \quad f_{\text{even}}(t) = \frac{1}{2}(f(t) + f(-t)), \quad f_{\text{odd}}(t) = \frac{1}{2}(f(t) - f(-t)).$$

Verify that $f_{\text{even}}(t)$ indeed is even, and that $f_{\text{odd}}(t)$ indeed is an odd function (regardless of $f(t)$).

Example 159. The **hyperbolic cosine**, denoted $\cosh(t)$, is the even part of e^t. Likewise, the **hyperbolic sine**, denoted $\sinh(t)$, is the odd part of e^t.

- Equivalently, $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$ and $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$.
- In particular, $e^t = \cosh(t) + \sinh(t)$.

  As recalled above, any function is the sum of its even and odd part.

  Comparing with Euler's formula, we find $\cosh(it) = \cos(t)$ and $\sinh(it) = i\sin(t)$. This indicates that $\cosh$ and $\sinh$ are related to $\cos$ and $\sin$, as their name suggests (see below for the "hyperbolic" part).
- $\frac{\mathrm{d}}{\mathrm{d}t}\cosh(t) = \sinh(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\sinh(t) = \cosh(t)$.
- $\cosh(t)$ and $\sinh(t)$ both satisfy the DE $y'' = y$.

  We can write the general solution as $C_1 e^t + C_2 e^{-t}$ or, if useful, as $C_1\cosh(t) + C_2\sinh(t)$.
- $\cosh(t)^2 - \sinh(t)^2 = 1$

  Verify this by substituting $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$ and $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$.

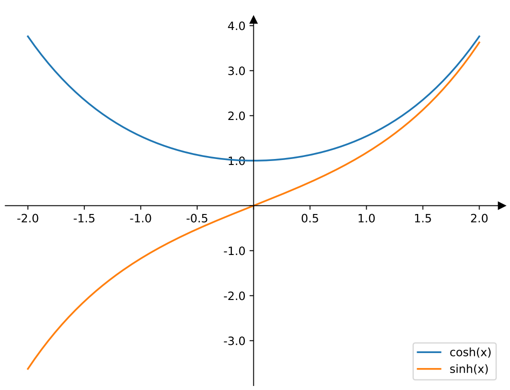
  Note that the equation $x^2 - y^2 = 1$ describes a **hyperbola** (just like $x^2 + y^2 = 1$ describes a circle).

  The above equation is saying that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cosh(t) \\ \sinh(t) \end{bmatrix}$ is a parametrization of the hyperbola.

  Comment. Circles and hyperbolas are conic sections (as are ellipses and parabolas).

  Comment. Hyperbolic geometry plays an important role, for instance, in special relativity:
  https://en.wikipedia.org/wiki/Hyperbolic_geometry

Homework. Write down the parallel properties of cosine and sine.



Armin Straub
straub@southalabama.edu

The fin equation from thermodynamics

The following is an example from thermodynamics. The governing differential equation is a second-order DE that is like the equation describing the motion of a mass on a spring ($my'' + ky = 0$) except that one term has the opposite sign. Besides showcasing an application, we want to show off how $\cosh$ and $\sinh$ are useful for writing certain solutions in a more pleasing form.

Let $T(x)$ describe the temperature at position x in a fin with fin base at $x = 0$ and fin tip at $x = L$.

For more context on fins: https://en.wikipedia.org/wiki/Fin_(extended_surface)

If we write $\theta(x) = T(x) - T_\infty$ for the temperature excess at position x (with T_∞ the external temperature), then we find (under various simplifying assumptions) that the temperature distribution in our fin satisfies the following DE, known as the **fin equation**:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - m^2\theta = 0, \quad m^2 = \frac{hP}{kA} > 0.$$

- A is the cross-sectional area of the fin (assumed to be the same for all positions x).
- P is the perimeter of the fin (assumed to be the same for all positions x).
- k is the thermal conductivity of the material (assumed to be constant).
- h is the convection heat transfer coefficient (assumed to be constant).

Since the DE is homogeneous and linear with characteristic roots $\pm m$, the general solution is

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} = D_1 \cosh(mx) + D_2 \sinh(mx).$$

The constants C_1, C_2 (or, equivalently, D_1, D_2) can then be found by emposing appropriate boundary conditions at the **fin base** ($x = 0$) and at the **fin tip** ($x = L$).

In practice, we often know the temperature at the fin base and therefore the temperature excess, resulting in the boundary condition $\theta(0) = \theta_0$. At the fin tip, common boundary conditions are:

- $\theta(L) \to 0$ as $L \to \infty$ (infinitely long fin)

  In this case, the fin is so long that the temperature at the fin tip approaches the external temperature. Mathematically, we get $\theta(x) = Ce^{-mx}$ since $e^{mx} \to \infty$ as $x \to \infty$. It follows from $\theta(0) = \theta_0$ that $C = \theta_0$.

  Thus, the temperature excess is $\theta(x) = \theta_0 e^{-mx}$.

- $\theta'(L) = 0$ (neglible heat loss at the fin tip, "adiabatic fin tip")

  This can be a more reasonable assumption than the infinitely long fin. Note that the total heat transfer from the fin is proportional to its surface area. If the surface area at the fin tip is a negligible fraction of the total surface area, then it is reasonable to assume that $\theta'(L) = 0$.

  In this case, the temperature excess is $\theta(x) = \theta_0 \frac{\cosh(m(L-x))}{\cosh(mL)}$.

  Check! Instead of computing this from scratch (do that as well, later!), check that this indeed solves the DE as well as the boundary conditions $\theta(0) = \theta_0$ and $\theta'(L) = 0$. This should be a rather quick check!

- $\theta(L) = \theta_L$ (specified temperature at fin tip)

  In this case, the temperature excess is $\theta(x) = \frac{\theta_L \sinh(mx) + \theta_0 \sinh(m(L-x))}{\sinh(mL)}$.

  Check! Again, check that this indeed solves the DE as well as the boundary conditions $\theta(0) = \theta_0$ and $\theta(L) = \theta_L$. Once more, this should be a quick and pleasant check.

Armin Straub
straub@southalabama.edu

Application to military strategy: Lanchester's equations

In military strategy, Lanchester's equations can be used to model two opposing forces during "aimed fire" battle.

Let $x(t)$ and $y(t)$ describe the number of troops on each side. Then Lanchester (during World War I) assumed that the rates $-x'(t)$ and $-y'(t)$, at which soldiers are put out of action, are proportional to the number of opposing forces. That is:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\beta y(t) \\ -\alpha x(t) \end{bmatrix}, \quad \text{or, in matrix form: } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -\beta \\ -\alpha & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The proportionality constants $\alpha, \beta > 0$ indicate the strength of the forces ("fighting effectiveness coefficients").

These are simple linear DEs with constant coefficients, which we have learned how to solve.

Comment. The "aimed fire" means that all combatants are engaged, as is common in modern combat with long-range weapons. This is rather different than ancient combat where soldiers were engaging one opponent at a time.

For more details, see: `https://en.wikipedia.org/wiki/Lanchester%27s_laws`

Example 160. Solve Lanchester's equations subject to the initial conditions $x(0) = x_0$ and $y(0) = y_0$.

Solution. (using Laplace transforms) $x' = -\beta y$ transforms into $sX - x_0 = -\beta Y$. Likewise, $y' = -\alpha x$ transforms into $sY - y_0 = -\alpha X$. The transformed equations are regular equations that we can solve for X and Y.

For instance, by the first equation, $Y = -\frac{1}{\beta}(sX - x_0)$.

Used in the second equation, we get $-\frac{s}{\beta}(sX - x_0) - y_0 = -\alpha X$ so that $(s^2 - \alpha\beta)X = sx_0 - \beta y_0$.

Hence, the system is solved by $X = \frac{sx_0 - \beta y_0}{s^2 - \alpha\beta}$ and $Y = -\frac{1}{\beta}(sX - x_0) = \frac{sy_0 - \alpha x_0}{s^2 - \alpha\beta}$.

As a final step, we need to take the inverse Laplace transform to get $x(t) = \mathcal{L}^{-1}(X(s))$ and $y(t) = \mathcal{L}^{-1}(Y(s))$.

Using partial fractions, $X(s) = \frac{sx_0 - \beta y_0}{(s - \sqrt{\alpha\beta})(s + \sqrt{\alpha\beta})} = \frac{A}{s - \sqrt{\alpha\beta}} + \frac{B}{s + \sqrt{\alpha\beta}}$ with

$$A = \left.\frac{sx_0 - \beta y_0}{s + \sqrt{\alpha\beta}}\right|_{s = \sqrt{\alpha\beta}} = \frac{\sqrt{\alpha\beta}\, x_0 - \beta y_0}{2\sqrt{\alpha\beta}} = \frac{1}{2}\left(x_0 - y_0\sqrt{\frac{\beta}{\alpha}}\right), \quad B = \left.\frac{sx_0 - \beta y_0}{s - \sqrt{\alpha\beta}}\right|_{s = -\sqrt{\alpha\beta}} = \frac{1}{2}\left(x_0 + y_0\sqrt{\frac{\beta}{\alpha}}\right).$$

It follows that $x(t) = Ae^{\sqrt{\alpha\beta}\,t} + Be^{-\sqrt{\alpha\beta}\,t}$. We obtain a similar formula for $y(t)$ (with x_0 and y_0 as well as α and β swapped for each other).

Solution. (without Laplace transforms) Our goal is to write down a single DE that only involves, say, $x(t)$.

From the first DE, we get $y(t) = -\frac{1}{\beta}x'(t)$. Hence, $y'(t) = -\frac{1}{\beta}x''(t)$. Using that in the second DE, we obtain $-\frac{1}{\beta}x''(t) = -\alpha x(t)$ or, equivalently, $x''(t) - \alpha\beta x(t) = 0$.

Observe that, since $y(t) = -\frac{1}{\beta}x'(t)$, the initial condition $y(0) = y_0$ translates into $x'(0) = -\beta y_0$.

The roots are $\pm r$ where $r = \sqrt{\alpha\beta}$. Hence, $x(t) = C_1 e^{rt} + C_2 e^{-rt}$.

Using the initial conditions $x(0) = x_0$ and $x'(0) = -\beta y_0$, we find $C_1 + C_2 = x_0$ and $rC_1 - rC_2 = -\beta y_0$.

This results in $C_1 = \frac{1}{2}\left(x_0 - \frac{\beta y_0}{r}\right)$ and $C_2 = \frac{1}{2}\left(x_0 + \frac{\beta y_0}{r}\right)$.

Correspondingly, using $r = \sqrt{\alpha\beta}$,

$$x(t) = \frac{1}{2}\left(x_0 - y_0\sqrt{\frac{\beta}{\alpha}}\right)e^{\sqrt{\alpha\beta}\,t} + \frac{1}{2}\left(x_0 + y_0\sqrt{\frac{\beta}{\alpha}}\right)e^{-\sqrt{\alpha\beta}\,t}$$

with a similar formula for $y(t) = -\frac{1}{\beta}x'(t)$.

Comment. The formulas take a particularly pleasing form when written in terms of cosh and sinh instead:

$$x(t) = x_0 \cosh(\sqrt{\alpha\beta}\, t) - y_0 \sqrt{\frac{\beta}{\alpha}} \sinh(\sqrt{\alpha\beta}\, t),$$
$$y(t) = y_0 \cosh(\sqrt{\alpha\beta}\, t) - x_0 \sqrt{\frac{\alpha}{\beta}} \sinh(\sqrt{\alpha\beta}\, t).$$

Example 161. Determine conditions on x_0, y_0 (size of forces) and α, β (effectiveness of forces) that allow us to conclude who will win the battle.

Solution. We analyze our explicit formulas to find out which of $x(t)$ and $y(t)$ becomes 0 first (and therefore loses the battle). Note that both solutions are combinations of $e^{\sqrt{\alpha\beta}\, t}$ and $e^{-\sqrt{\alpha\beta}\, t}$. Further note that the term $e^{\sqrt{\alpha\beta}\, t}$ dominates the other as t gets large.

Since $y(t) = -\frac{1}{\beta}x'(t)$, the coefficients of $e^{\sqrt{\alpha\beta}\, t}$ in the two solutions $x(t)$ and $y(t)$ have opposite signs (for $x(t)$ that coefficient is $\frac{1}{2}(x_0 - y_0\sqrt{\beta/\alpha})$). This allows us to conclude that $x(t)$ wins the battle if $x_0 - y_0\sqrt{\frac{\beta}{\alpha}} > 0$. This is equivalent to $\alpha x_0^2 > \beta y_0^2$.

Solution. (without solving the DE) As an alternative, we can also start fresh and divide the two equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\beta y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -\alpha x$$

to get $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\alpha x}{\beta y}$. Using separation of variables, we find $\beta y\, \mathrm{d}y = \alpha x\, \mathrm{d}x$ which implies $\frac{1}{2}\beta y^2 = \frac{1}{2}\alpha x^2 + D$.

Consequently, $\alpha x^2 - \beta y^2 = C$ where $C = -2D$ is a constant. Using the initial conditions, we find $C = \alpha x_0^2 - \beta y_0^2$.
If $y(t)$ is zero first (x wins), then $\alpha x(t)^2 = C > 0$. On the other hand, if $x(t)$ is zero first, then $-\beta y(t)^2 = C < 0$. In other words, the sign of C determines who will win the battle.
Namely, x will win if $C > 0$ which is equivalent to $\alpha x_0^2 > \beta y_0^2$.

Conclusion. The condition we found is known as **Lanchester's square law**: its crucial message is that the sizes x_0, y_0 of the forces count quadratically, whereas the fighting effectivenesses α, β only count linearly. In other words, to beat a force with twice the effectiveness the other side only needs to have a force that is about 41.4% larger (since $\sqrt{2} \approx 1.4142$). Or, put differently, to beat a force of twice the size, the other side would need a fighting effectiveness that is more than 4 times as large.