Example 140. (review) Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right)$.

Solution. Note that $s^2-s-6=(s-3)(s+2)$. We use **partial fractions** to write $-\frac{6s-23}{(s-3)(s+2)}=\frac{A}{s-3}+\frac{B}{s+2}$. We find the coefficients (see brief review below) as

$$A=-\frac{6s-23}{s+2}\bigg|_{s=3}=1,\quad B=-\frac{6s-23}{s-3}\bigg|_{s=-2}=-7.$$

Hence $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right)=\mathcal{L}^{-1}\left(\frac{1}{s-3}-\frac{7}{s+2}\right)=\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)-7\mathcal{L}^{-1}\left(\frac{7}{s+2}\right)=e^{3t}-7e^{-2t}$.

Review. In order to find A, we multiply $-\frac{6s-23}{(s-3)(s+2)}=\frac{A}{s-3}+\frac{B}{s+2}$ by $s-3$ to get $-\frac{6s-23}{s+2}=A+\frac{B(s-3)}{s+2}$. We then set $s=3$ to find A as above.

Comment. Compare with Example 131 where we considered the same functions.

Example 141. Solve the IVP $y''-3y'+2y=e^{-t}$, $y(0)=0$, $y'(0)=1$.

Solution. (old style) The characteristic polynomial $D^2-3D+2=(D-1)(D-2)$ has ("old") roots $1,2$. The "new" root is -1. Since there is no duplication, there must be a particular solution of the form $y_p(t)=Ae^{-t}$.

To determine A, we plug into the DE $y_p''-3y_p'+2y_p=6Ae^{-t}\overset{!}{=}e^{-t}$ and conclude $A=\frac{1}{6}$.

The general solution thus is $y(t)=\frac{1}{6}e^{-t}+C_1e^t+C_2e^{2t}$. We need to find C_1 and C_2 using the initial conditions.

Solving $y(0)=\frac{1}{6}+C_1+C_2\overset{!}{=}0$ and $y'(0)=-\frac{1}{6}+C_1+2C_2\overset{!}{=}1$, we find $C_2=\frac{4}{3}$ and $C_1=-\frac{3}{2}$.

Hence, the unique solution to the IVP is $y(t)=\frac{1}{6}e^{-t}-\frac{3}{2}e^t+\frac{4}{3}e^{2t}$.

Solution. (Laplace style) The differential equation (plus initial conditions!) transforms as follows:

$$\begin{aligned}
\mathcal{L}(y''(t))-3\,\mathcal{L}(y'(t))+2\mathcal{L}(y(t)) &= \mathcal{L}(e^{-t})\\
s^2Y(s)-sy(0)-y'(0)-3(sY(s)-y(0))+2Y(s) &= \frac{1}{s+1}\\
(s^2-3s+2)Y(s) &= 1+\frac{1}{s+1}=\frac{s+2}{s+1}\\
Y(s) &= \frac{s+2}{(s^2-3s+2)(s+1)}\\
&= \frac{s+2}{(s-1)(s-2)(s+1)}
\end{aligned}$$

To find $y(t)$, we use partial fractions to write $Y(s)=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+1}$. We find the coefficients as

$$A=\frac{s+2}{(s-2)(s+1)}\bigg|_{s=1}=-\frac{3}{2},\quad B=\frac{s+2}{(s-1)(s+1)}\bigg|_{s=2}=\frac{4}{3},\quad C=\frac{s+2}{(s-1)(s-2)}\bigg|_{s=-1}=\frac{1}{6}.$$

Hence, $y(t)=\mathcal{L}^{-1}\left(\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+1}\right)=Ae^t+Be^{2t}+Ce^{-t}=\frac{1}{6}e^{-t}-\frac{3}{2}e^t+\frac{4}{3}e^{2t}$, as above.

Comment. Note the factor s^2-3s+2 in front of $Y(s)$ when we transformed the DE. This is the characteristic polynomial. Can you see how the "old" and "new" roots show up in the Laplace transform approach?

Example 142. Consider the IVP $y''-3y'+2y=e^{-t}$, $y(0)=0$, $y'(0)=1$.

Determine the Laplace transform of the unique solution.

Solution. We just did that! By transforming the DE, we found that $Y(s)=\frac{s+2}{(s-1)(s-2)(s+1)}$.

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Example 143. Consider the IVP $y'' - 3y' + y = 2e^{5t}$, $y(0) = -1$, $y'(0) = 4$.

Determine the Laplace transform of the unique solution.

Solution. The DE $y'' - 3y' + y = 2e^{5t}$ (plus initial conditions!) transforms into

$$s^2 Y - s y(0) - y'(0) - 3(sY - y(0)) + Y = (s^2 - 3s + 1)Y + (s - 7) = \frac{2}{s-5}.$$

Accordingly, $Y(s) = \frac{1}{s^2 - 3s + 1}\left[\frac{2}{s-5} - s + 7\right]$ is the Laplace transform of the unique solution to the IVP.

Comment. The characteristic roots are $(3 \pm \sqrt{5})/2$. As a result, the solution $y(t)$ will be rather unpleasant to write down by hand, with coefficients that are not rational numbers. By contrast, the above Laplace transform can be expressed without irrational numbers.

Comment. Depending on what we intend to do with the solution, we might not even need $y(t)$ but might instead be able to extract what we want from its Laplace transform $Y(s)$.

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