Some important differential operators: ∇f, $\nabla \cdot \boldsymbol{F}$, $\nabla \times \boldsymbol{F}$, $\nabla^2 f$

An **operator** takes a function as input, and returns another function as output.

An example of an operator is the derivative $\frac{\mathrm{d}}{\mathrm{d}x}$. A more recent example is ∇.

The **divergence** of $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{k}$ is $\operatorname{div} \boldsymbol{F} = \nabla \cdot F = M_x + N_y + P_z$.

- If we think of $\boldsymbol{F}$ as the velocity of some fluid (or gas) flowing, then $\operatorname{div} \boldsymbol{F}$ measures whether the fluid is expanding (or shrinking) at any given point. In other words, imagine a tiny box around a point; $\operatorname{div} \boldsymbol{F}$ will be the net amount of stuff flowing into that box (stuff flowing into, minus stuff flowing out). See Figure 15.26 and the accompanying explanation for why this is measured by the formula above.
- Real fluids (but not gases) are often **incompressible**, which means that $\operatorname{div} \boldsymbol{F} = 0$ because there is as much flowing into the tiny boxes as there is flowing out.

The **curl** of $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{k}$ is $\operatorname{curl} \boldsymbol{F} = \nabla \times F = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} M \\ N \\ P \end{bmatrix} = \begin{bmatrix} P_y - N_z \\ M_z - P_x \\ N_x - M_y \end{bmatrix}$.

- Note that the component test says: $\boldsymbol{F}$ is conservative on a simply connected region iff $\operatorname{curl} \boldsymbol{F} = \boldsymbol{0}$.
- The curl measures microscopic circulation: Imagine a tiny sphere immersed in our fluid, with its center fixed at a point. It might start spinning due to the flow; the axis of this rotation is $\operatorname{curl} \boldsymbol{F}$ and the magnitude of $\operatorname{curl} \boldsymbol{F}$ indicates the speed of rotation. Figure 15.29 shows, in the 2D case, how to connect this interpretation with the formula above.

Example 152. Compute $\operatorname{div} \boldsymbol{F}$ and $\operatorname{curl} \boldsymbol{F}$ for **(a)** $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j}$ and **(b)** $\boldsymbol{F} = -y\boldsymbol{i} + x\boldsymbol{j}$.

Solution. Recall that we sketched both of these fields before!

(a) We find $\operatorname{div} \boldsymbol{F} = 2$ and $\operatorname{curl} \boldsymbol{F} = \boldsymbol{0}$.

Comment. $\operatorname{curl} \boldsymbol{F} = \boldsymbol{0}$ is also a consequence of the fact that we already know that $\boldsymbol{F}$ is conservative (it has potential function $f(x, y, z) = \frac{x^2}{2} + \frac{y^2}{2}$).

Comment. Try to see from the sketch of the "radial field" $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j}$ that there is more flowing out of each tiny box than there is flowing in. That's because stuff is flowing away from the origin and speeding up with increasing distance. Hence, while there is stuff flowing into the box on its side closer to the origin, there is even more flowing out on the side further away from the origin.

Can you also see from the sketch (by symmetry considerations) why $\operatorname{curl} \boldsymbol{F} = \boldsymbol{0}$?

(b) $\operatorname{div} \boldsymbol{F} = 0$ and $\operatorname{curl} \boldsymbol{F} = 2\boldsymbol{k}$.

Can you make sense of these values from the sketch of this "spin field"?

Example 153. Compute $\operatorname{div} \boldsymbol{F}$, $\operatorname{curl} \boldsymbol{F}$ and $\operatorname{div} \operatorname{curl} \boldsymbol{F}$ for $\boldsymbol{F} = \begin{bmatrix} xy \\ y^2 \\ x - z \end{bmatrix}$.

Solution. $\operatorname{div} \boldsymbol{F} = y + 2y - 1$, $\operatorname{curl} \boldsymbol{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} xy \\ y^2 \\ x - z \end{bmatrix} = \begin{bmatrix} 0 - 0 \\ 0 - 1 \\ 0 - x \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -x \end{bmatrix}$, $\operatorname{div} \operatorname{curl} \boldsymbol{F} = 0 + 0 + 0 = 0$.

Comment. In fact, we always have $\operatorname{div} \operatorname{curl} \boldsymbol{F} = 0$. Do you see how to show that?

Example 154. Spell out the differential operator $\nabla \cdot \nabla$, which is commonly abbreviated as ∇^2.

Solution. $\nabla^2 f = \nabla \cdot \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = f_{xx} + f_{yy} + f_{zz}$. ∇^2 is known as the **Laplace operator**.

Comment. We briefly looked at the partial differential equation $\nabla^2 f = 0$ in Example 82.

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