Review. Is $\boldsymbol{F}$ conservative on a **simply connected** region D? (In 2D: roughly, a region without "holes".)

- $\boldsymbol{F}(x,y) = \begin{bmatrix} M(x,y) \\ N(x,y) \end{bmatrix}$ is conservative if and only if $M_y = N_x$.
- $\boldsymbol{F}(x,y,z) = \begin{bmatrix} M(x,y,z) \\ N(x,y,z) \\ P(x,y,z) \end{bmatrix}$ is conservative if and only if $M_y = N_x$, $M_z = P_x$, $N_z = P_y$.

Review. Are the vector fields $\boldsymbol{F} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\boldsymbol{G} = \begin{bmatrix} -y \\ x \end{bmatrix}$ conservative?

Solution. (component test) We check whether $M_y = N_x$.

(a) $M_y = \frac{\partial}{\partial y} x = 0$ and $N_x = \frac{\partial}{\partial x} y = 0$ are equal. Hence, $\boldsymbol{F}$ is conservative on $\mathbb{R}^2$ (which is simply connected).

(b) $M_y = \frac{\partial}{\partial y}(-y) = -1$ and $N_x = \frac{\partial}{\partial x} x = 1$ are not equal. Hence, $\boldsymbol{G}$ is not conservative.

Example 144. Is the vector field $\boldsymbol{F} = (2x+z)\boldsymbol{i} - 2yz\boldsymbol{j} + (x - y^2 + 1)\boldsymbol{k}$ conservative? If so, determine a potential function.

Solution. We check whether $M_y = N_x$, $M_z = P_x$, $N_z = P_y$. Indeed, $M_y = N_x = 0$, $M_z = P_x = 1$, $N_z = P_y = -2y$ are all equal. Hence, $\boldsymbol{F}$ is conservative on $\mathbb{R}^3$ (which is simply connected).

To find a potential function, we start with $f_x = 2x + z$ (or one of the other three equations). This implies that $f = x^2 + xz + C(y,z)$. Then, comparing derivatives with respect to y, $f_y = C_y(y,z) = -2yz$ and hence $C(y,z) = -y^2 z + D(z)$. So far, we found that $f = x^2 + xz - y^2 z + D(z)$. Finally, comparing derivatives with respect to z, $f_z = x - y^2 + D_z(z) = x - y^2 + 1$. Simplified, this is $D_z = 1$, so $D = z + E$.

In conclusion, we found the potential function $f(x,y,z) = x^2 + xz - y^2 z + z$.

[The general potential function is $f(x,y,z) = x^2 + xz - y^2 z + z + E$. Just as for ordinary antiderivatives, a constant can be added to any particular potential function to produce all the others.]

Example 145. Consider the vector field $\boldsymbol{F} = \frac{-y}{x^2+y^2}\boldsymbol{i} + \frac{x}{x^2+y^2}\boldsymbol{j}$.

(a) What is the natural domain of $\boldsymbol{F}$? Is this set simply connected?

(b) Compute (do the integral!) $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ where C is the unit circle, traversed counterclockwise from $(1,0)$.

(c) Is the vector field $\boldsymbol{F}$ conservative on its natural domain?

(d) Is the vector field $\boldsymbol{F}$ conservative on the region defined by $y > 0$?

(e) What is $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ if C is the circle of radius 1 around $(0,2)$?

Solution.

(a) The natural domain is all of the plane $\mathbb{R}^2$, with the exception of $(0,0)$.
This set is not simply connected because of the "hole" at the origin.

(b) Do it! The final answer is $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 2\pi$.

(c) $\boldsymbol{F}$ is not conservative on its natural domain. If it was, then the previous integral would have to be 0. (Why?)

(d) The region defined by $y > 0$ is simply connected. Hence, we can use the component test to determine whether $\boldsymbol{F}$ is conservative on this region. Do it! In the end, that $\boldsymbol{F}$ is conservative there.

(e) Note that C is contained in the region defined by $y > 0$. Since $\boldsymbol{F}$ is conservative there, $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0$.
[It would be much more work to compute the integral directly.]

Comment. Since $\boldsymbol{F}$ is conservative on regions excluding $(0,0)$ without holes, it has a potential function there. Indeed, if $x > 0$, then $\boldsymbol{F}$ is the gradient field of $f(x,y) = \arctan(y/x)$. Check that! Note that $f(x,y) = \theta$ where θ is the angle used in polar coordinates. (Why?) Can you see how this explains the answer 2π we got in the second part? (In a way, the integral is still $f(\text{end}) - f(\text{start})$ but $f(\text{start}) = \theta_{\text{start}} = 0$ and $f(\text{end}) = \theta_{\text{end}} = 2\pi$, even though start and end are the same point. f is "multi-valued", something we don't allow for ordinary functions. This is avoided by requiring the region in question to be simply connected.)

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