Example 56. (intersecting two planes) Determine the line of intersection of the two planes

$$x + 2y + z = 2, \qquad x + y - 2z = 1.$$

Solution. Subtracting the second equation from the first (to eliminate x), we find $y + 3z = 1$. We now choose to set $z = t$, our parameter. From $y + 3t = 1$, we then find $y = 1 - 3t$.

Substituting into the first equation, we get $x + 2(1 - 3t) + t = 2$, which gives $x = 5t$.

Taken together, we have $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5t \\ 1-3t \\ t \end{bmatrix}$. This is a parametrization of the line, with direction vector $\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$.

Exercise. Repeat the computation but choose to set $y = t$. What changes?

[You will get a different parametrization but of the same line. Compare the direction vectors!]

Example 57. (intersecting two planes, again) Can we find the direction vector directly?

Solution. Note that $\boldsymbol{n}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{n}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ are the normal vectors of the two planes.

$\boldsymbol{n}_1$ is perpendicular to plane #1. In particular, $\boldsymbol{n}_1$ is perpendicular to the line of intersection.

Likewise, $\boldsymbol{n}_2$ is perpendicular to plane #2. In particular, $\boldsymbol{n}_2$ is perpendicular to the line of intersection.

Hence, the direction vector of our line is perpendicular to both $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, and we can compute this direction as $\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ -1 \end{bmatrix}$.

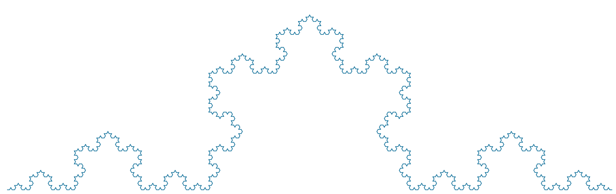
Note. The direction vector is the same as before, up to scaling (here, a factor of -1). The direction of the line is unique (but not the length or orientation of the direction vector we choose).

Just for fun!

Why are perimeters of countries missing from wikipedia?

Or, why is the coastline of the UK listed as 11,000 miles by the UK mapping authority but 7,700 miles by the CIA Factbook?

Koch snowflake, fractals, 1.26 dimensions, $\ldots$



Some of the fun can be found at:

`https://en.wikipedia.org/wiki/Coastline_paradox`

Armin Straub
straub@southalabama.edu