Preparing for Midterm #2

MATH 126 — Calculus II
Friday, Nov 13

Please print your name:

Problem 1. Go over all the quizzes!

To help you with that, there is a version of each quiz posted on our course website without solutions (of course, there's solutions, too).

Problem 2. Determine the following limits.

(a) $\lim\limits_{n\to\infty} \dfrac{1+\log(n)}{1+n^{3/2}}$

(b) $\lim\limits_{n\to\infty} \dfrac{5^n+3^n}{4^n-1}$

(c) $\lim\limits_{n\to\infty} \left(1-\dfrac{1}{2n}\right)^n$

Solution.

(a) $\lim\limits_{n\to\infty} \dfrac{1+\log(n)}{1+n^{3/2}} = 0$ because $n^{3/2}$ grows much faster than $\log(n)$.

If we want to be very precise, we can apply L'Hospital to arrive at the same conclusion.

(b) $\lim\limits_{n\to\infty} \dfrac{5^n+3^n}{4^n-1} = \infty$

You should be able to see this limit because 5^n outgrows all the other terms.

To make our "vision" precise, we can divide top and bottom by 5^n (that doesn't change anything):

$$\lim_{n\to\infty} \frac{5^n+3^n}{4^n-1} = \lim_{n\to\infty} \frac{1+\left(\frac{3}{5}\right)^n}{\left(\frac{4}{5}\right)^n - \frac{1}{5^n}} = \infty$$

because $\left(\frac{3}{5}\right)^n$, $\left(\frac{4}{5}\right)^n$ and $\frac{1}{5^n}$ all approach 0 as $n\to\infty$.

(c) We apply log to the sequence to get $\log\left(\left(1-\frac{1}{2n}\right)^n\right) = n\log\left(1-\frac{1}{2n}\right) = \frac{\log\left(1-\frac{1}{2n}\right)}{\frac{1}{n}}$. After the last step, we have a quotient in indeterminate form "$\frac{0}{0}$", and so can apply L'Hospital to find

$$\lim_{n\to\infty} \frac{\log\left(1-\frac{1}{2n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{1-\frac{1}{2n}}\cdot\frac{1}{2n^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \left(-\frac{1}{2}\frac{1}{1-\frac{1}{2n}}\right) = -\frac{1}{2}.$$

But recall that this is not $L = \lim\limits_{n\to\infty} \left(1-\dfrac{1}{2n}\right)^n$ but instead we found $\log(L) = -\frac{1}{2}$.

Our original limit therefore is $\lim\limits_{n\to\infty} \left(1-\dfrac{1}{2n}\right)^n = e^{-1/2}$. □

Problem 3. Suppose that $\lim\limits_{n\to\infty} a_n = L$.

(a) Determine: $\lim\limits_{n\to\infty} a_n^2$

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(b) Determine: $\lim_{n\to\infty} a_{n^2}$

(c) Suppose further that $a_{n+1} = 2 - \frac{a_n}{3}$. What is L?

Solution.

(a) $\lim_{n\to\infty} a_n^2 = L^2$

(b) $\lim_{n\to\infty} a_{n^2} = L$

(c) Taking limits on both sides of $a_{n+1} = 2 - \frac{a_n}{3}$, we get $L = 2 - \frac{L}{3}$, which we solve to find $L = \frac{3}{2}$. □

Problem 4. Determine whether the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{7\sqrt{n} + \log(n)}{n^2+4}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{10\log(n)}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n-4}{n^2+\log(n)}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^n}{7n^2+1}$

Solution.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{7\sqrt{n} + \log(n)}{n^2+4}$ converges. (Limit comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which converges).

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{10\log(n)}$ diverges because the terms $\displaystyle\frac{\sqrt{n}}{10\log(n)}$ do not converge to 0 as $n\to\infty$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n-4}{n^2+\log(n)}$ diverges. (Limit comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges).

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^n}{7n^2+1}$ diverges because the terms $\displaystyle\frac{(-4)^n}{7n^2+1}$ do not converge to 0 as $n\to\infty$. □

Problem 5. Using the integral test, determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{3/2}}$ converges.

Solution. By the integral test, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{3/2}}$ converges if and only if the integral $\displaystyle\int_2^{\infty} \frac{\mathrm{d}x}{x(\log x)^{3/2}}$ converges.

First, however, we should verify that the integral test indeed applies: the function $\displaystyle\frac{1}{x(\log x)^{3/2}}$ is obviously positive and continuous for $x \geqslant 2$. It is also decreasing, because $x(\log x)^{3/2}$ clearly increases.

Upon substituting $u = \log x$, we find that

$$\int_2^{\infty} \frac{\mathrm{d}x}{x(\log x)^{3/2}} = \int_{\log(2)}^{\infty} \frac{\mathrm{d}u}{u^{3/2}} = \left[-\frac{2}{u^{1/2}}\right]_{\log(2)}^{\infty}$$

is finite because $\lim_{u\to\infty}\left(-\frac{2}{u^{1/2}}\right)=0$. Therefore, the series $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^{3/2}}$ converges. ☐

Problem 6. For which values of x does $\sum_{n=2}^{\infty}\frac{(-1)^n x^n+1}{3^n}$ converge?

Evaluate the series (as a function of x) for these values.

Solution.

$$\begin{aligned}
\sum_{n=2}^{\infty}\frac{(-1)^n x^n+1}{3^n} &= \sum_{n=2}^{\infty}\frac{(-1)^n x^n}{3^n}+\sum_{n=2}^{\infty}\frac{1}{3^n}\\
&= \sum_{n=2}^{\infty}\left(-\frac{x}{3}\right)^n+\sum_{n=2}^{\infty}\left(\frac{1}{3}\right)^n\\
&= \sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n-1+\frac{x}{3}+\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n-1-\frac{1}{3}\\
[\text{if } |-x/3|<1] \quad &= \frac{1}{1-(-\frac{x}{3})}-1+\frac{x}{3}+\frac{1}{1-\frac{1}{3}}-1-\frac{1}{3}\\
&= \frac{3}{3+x}+\frac{x}{3}-\frac{5}{6}
\end{aligned}$$

In particular, the series converges provided that $|-x/3|<1$, or, equivalently, $|x|<3$.

[You could have also used that ratio test to determine convergence. In this case, this is really not necessary because we can see that the series is a combination of two geometric series.] ☐

Problem 7. Determine the radius of convergence of the following power series.

(a) $\sum_{n=2}^{\infty}\frac{n!(x+1)^n}{10^n}$

(b) $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n3^n}$

(c) $\sum_{n=0}^{\infty}\binom{2n}{n}x^n$

Recall that $\binom{2n}{n}=\frac{(2n)!}{n!n!}$.

Solution.

(a) We apply the ratio test with $a_n=\frac{n!(x+1)^n}{10^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{(n+1)!(x+1)^{n+1}}{10^{n+1}}\cdot\frac{10^n}{n!(x+1)^n}\right|=\left|\frac{x+1}{10}\right|\frac{(n+1)!}{n!}=\left|\frac{x+1}{10}\right|(n+1)\to\infty \text{ as } n\to\infty \text{ (unless } x=-1)$$

The ratio test implies that $\sum_{n=2}^{\infty}\frac{n!(x+1)^n}{10^n}$ diverges if $x\neq-1$.

The radius of convergence therefore is 0.

(b) We apply the ratio test with $a_n=\frac{(x-2)^n}{n3^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{(x-2)^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{(x-2)^n}\right|=\left|\frac{x-2}{3}\right|\frac{n}{n+1}\to\left|\frac{x-2}{3}\right| \text{ as } n\to\infty$$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n 3^n}$ converges if $\left|\frac{x-2}{3}\right| < 1$ or, equivalently, $|x-2| < 3$.

The radius of convergence therefore is 3.

(c) We apply the ratio test with $a_n = \frac{(2n)!}{n!n!} x^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2n+2)!x^{n+1}}{(n+1)!(n+1)!} \frac{n!n!}{(2n)!x^n}\right| = |x| \frac{(2n+1)(2n+2)}{(n+1)(n+1)} \to 4|x| \text{ as } n \to \infty$$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!} x^n$ converges if $|x| < 1/4$. The radius of convergence is $1/4$. □

Problem 8.

(a) Integrate both sides of $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. By choosing the appropriate constant of integration, find a power series for $\log(1-x)$.

(b) What is the radius of convergence of this power series?

(c) Does the power series converge for $x = 1$? On the other hand, it turns out that it does converge for $x = -1$. Write down the first few terms of the series in the case $x = -1$.

Solution.

(a) First, note that $\int \frac{\mathrm{d}x}{1-x} = -\log(1-x) + C$.

Integrating the power series term by term thus yields $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\log(1-x) + C$.

Setting $x = 0$, we find that $C = 0$.

Therefore, $\log(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^n}{n}$.

(b) The geometric series converges for $|x| < 1$, and so has radius of convergence 1. This does not change when integrating (or differentiating) the series term by term.

Alternatively, you can see that the radius of convergence is still 1 by applying the ratio test.

(c) For $x = 1$, the series $-\sum_{n=1}^{\infty} \frac{x^n}{n}$ becomes the harmonic series, and so diverges.

For $x = -1$, the series is $-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$ is the alternating harmonic series.

This series converges (we haven't discussed yet, why) and $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \log(1-(-1)) = \log(2)$. □

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