Parametric curves

Example 213. The unit circle is described by the cartesian equation $x^2 + y^2 = 1$ (in polar coordinates, the equation would be $r = 1$). Instead of such coordinate equations, we can also describe the same curve by **parametrizing** it: $x = \cos\theta$, $y = \sin\theta$ with parameter $\theta \in [0, 2\pi]$.

Note. A curve can be parametrized in many ways. For instance, $x = t$, $y = \sqrt{1 - t^2}$ with $t \in [-1, 1]$ is another parametrization of the upper half-circle.

Remark. Note the difference in philosophies behind describing curves: an equation like $x^2 + y^2 = 1$ is more "exclusionary" because we start with all points (x, y) and then restrict to those with $x^2 + y^2 = 1$. On the other hand, $x = \cos\theta$, $y = \sin\theta$ with $\theta \in [0, 2\pi]$ is more "inclusionary" because we are listing only precisely the points on the curve.

Example 214. Find the slope of the line tangent to the curve $x = \cos\theta$, $y = \sin\theta$ at the point corresponding to $\theta = \frac{\pi}{4}$.

Solution. Obviously, the answer is that the slope is -1. But let us go through a possible computation using the parametrization (because, of course, we cannot "see" the answer in more complicated cases).

Solution. The tangent line has slope $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{\cos\theta}{-\sin\theta}$, which for $\theta = \frac{\pi}{4}$ is $\frac{\cos\frac{\pi}{4}}{-\sin\frac{\pi}{4}} = -1$.

Note. Of course, we can also work from the other descriptions of the circle (such as $x^2 + y^2 = 1$). Do it!

We can work with parametric curves similarly to what we have been doing. For instance:

The parametric curve $x = f(t)$, $y = g(t)$ with $t \in [a, b]$ has **arc length**

$$L = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\,\mathrm{d}t.$$

Note that $\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}$ equals $\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}$ from earlier.

Example 215. Using the parametric curve $x = r\cos\theta$, $y = r\sin\theta$ with parameter $\theta \in [0, 2\pi]$, find the circumference of a circle of radius r.

Solution. $L = \int_0^{2\pi} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_0^{2\pi} \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}\,\mathrm{d}t = \int_0^{2\pi} r\,\mathrm{d}t = 2\pi r$

Example 216. What is the difference between the parametric curves

$$x = \cos\theta,\ y = \sin\theta \text{ with } \theta \in [0, 2\pi] \quad \text{and} \quad x = \cos(2\pi\theta),\ y = \sin(2\pi\theta) \text{ with } \theta \in [0, 1]?$$

Solution. It is the same curve (unit circle around the origin). Only the parametrization differs: in the second case, we "go through" the circle at 2π times the speed compared to the first case.

Example 217. Describe the parametric curve given by $x = t^2$, $y = t + 1$ with $t \geqslant 0$.

Solution. If $x = t^2$, then $t = \sqrt{x}$, and so the curve is given by the cartesian equation $y = \sqrt{x} + 1$.

Note. In general, eliminating the parameter, as we did here, may be very difficult or impossible.

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