Example 165. Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Solution. We apply the ratio test with $a_n = \frac{n^2}{2^n}$.

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{2^{n+1}} \frac{2^n}{n^2} = \frac{1}{2}\frac{(n+1)^2}{n^2} = \frac{1}{2}\frac{n^2+2n+1}{n^2} \to \frac{1}{2}$ as $n \to \infty$

Since $\frac{1}{2} < 1$, the ratio test implies that $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

Solution. Note that $\frac{2^n}{n^3} \to \infty \neq 0$. Hence, the series diverges.

Solution. We apply the ratio test with $a_n = \frac{2^n}{n^3}$.

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)^3} \frac{n^3}{2^n} = 2\frac{n^3}{(n+1)^3} = 2\frac{n^3}{n^3+3n^2+3n+1} \to 2$ as $n \to \infty$

Since $2 > 1$, the ratio test implies that $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges.

Power series

Definition 166. A **power series** (about $x = 0$) is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

More generally, a **power series** about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

Example 167. Investigate convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$

Solution. Of course, we know that the harmonic series diverges. If we apply the ratio test with $a_n = \frac{1}{n}$, then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1} \to 1$ as $n \to \infty$. Therefore, the ratio test is useless in this case.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n}$

Solution. We apply the ratio test with $a_n = \frac{x^n}{n}$.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{n+1}\frac{n}{x^n}\right| = |x|\frac{n}{n+1} \to |x|$ as $n \to \infty$

The ratio test implies that $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges if $|x| < 1$ (and diverges if $|x| > 1$).

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