Example 160. Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{2n+1}{3n^2+3}$

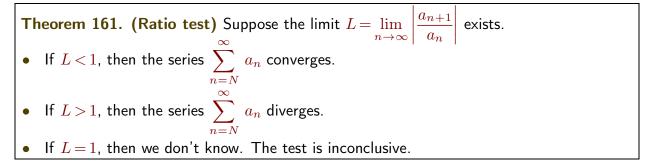
Compare the sequence $a_n = \frac{2n+1}{3n^2+3}$ to $b_n = \frac{1}{n}$ to conclude that both series diverge. [First, we check that $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{2}{3}$. By the limit comparison, we then find that $\sum_{n=1}^{\infty} \frac{2n+1}{3n^2+3}$ diverges because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.] (b) $\sum_{n=1}^{\infty} \frac{2\sqrt{n}+1}{3n^2+3}$

Compare this sequence to $b_n = \frac{1}{n^{3/2}}$ to conclude that both series converge.

Remark. The Riemann zeta function is defined by the sum $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$, which we know converges for p > 1. For complex values of $p \neq 1$, there is a unique way to "analytically continue" this function. It is then "easy" to see that $\zeta(-2) = 0$, $\zeta(-4) = 0$, The **Riemann hypothesis** claims that all other zeroes of $\zeta(p)$ lie on the line $p = \frac{1}{2} + a\sqrt{-1}$ ($a \in \mathbb{R}$). A proof of this conjecture (checked for the first 10,000,000,000 zeroes) is worth¹ \$1,000,000.

The ratio test

Note that a series $\sum_{n=N}^{\infty} a_n$ is geometric if $\frac{a_{n+1}}{a_n} = L$ is constant. It converges if and only if |L| < 1.



Example 162. Using the ratio test, analyze convergence of the geometric series $\sum_{n=0}^{\infty} x^n$.

Solution. In this case, $a_n = x^n$ and so $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{x^n}\right| = |x|.$

Therefore, the ratio test with $L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x|$ determines that $\sum_{n=0}^{\infty} x^n$ converges if |x| < 1, and diverges if |x| > 1.

[The ratio test makes no statement about the cases x = 1 and x = -1. We can check directly that the series diverges in both cases.]

^{1.} http://www.claymath.org/millenium-problems/riemann-hypothesis