Example 160. Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{3n^2+3}$

Compare the sequence $a_n = \frac{2n+1}{3n^2+3}$ to $b_n = \frac{1}{n}$ to conclude that both series diverge.

[First, we check that $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{2}{3}$. By the limit comparison, we then find that $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{3n^2+3}$ diverges because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.]

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2\sqrt{n}+1}{3n^2+3}$

Compare this sequence to $b_n = \frac{1}{n^{3/2}}$ to conclude that both series converge.

Remark. The Riemann zeta function is defined by the sum $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$, which we know converges for $p > 1$. For complex values of $p \neq 1$, there is a unique way to "analytically continue" this function. It is then "easy" to see that $\zeta(-2) = 0$, $\zeta(-4) = 0$, ... The **Riemann hypothesis** claims that all other zeroes of $\zeta(p)$ lie on the line $p = \frac{1}{2} + a\sqrt{-1}$ $(a \in \mathbb{R})$. A proof of this conjecture (checked for the first 10,000,000,000 zeroes) is worth[1] \$1,000,000.

The ratio test

Note that a series $\displaystyle\sum_{n=N}^{\infty} a_n$ is geometric if $\frac{a_{n+1}}{a_n} = L$ is constant. It converges if and only if $|L| < 1$.

Theorem 161. (Ratio test) Suppose the limit $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists.

- If $L < 1$, then the series $\displaystyle\sum_{n=N}^{\infty} a_n$ converges.
- If $L > 1$, then the series $\displaystyle\sum_{n=N}^{\infty} a_n$ diverges.
- If $L = 1$, then we don't know. The test is inconclusive.

Example 162. Using the ratio test, analyze convergence of the geometric series $\displaystyle\sum_{n=0}^{\infty} x^n$.

Solution. In this case, $a_n = x^n$ and so $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{x^n}\right| = |x|$.

Therefore, the ratio test with $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x|$ determines that $\displaystyle\sum_{n=0}^{\infty} x^n$ converges if $|x| < 1$, and diverges if $|x| > 1$.

[The ratio test makes no statement about the cases $x = 1$ and $x = -1$. We can check directly that the series diverges in both cases.]

1. `http://www.claymath.org/millenium-problems/riemann-hypothesis`

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