A few more examples of sequences:

- $\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$

  This is the sequence $\{a_n\}$ with $a_n = \frac{1}{n^2}$. Clearly, $\lim_{n\to\infty} a_n = 0$.

  We will learn later that the **series** $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots$ also converges and equals $\frac{\pi^2}{6}$.

- $1, 1, 2, 3, 5, 8, 13, 21, \ldots$

  These are the **Fibonacci numbers** $\{F_n\}$. They are defined **recursively**: $F_n = F_{n-1} + F_{n-2}$ together with the **initial values** $F_1 = 1$, $F_2 = 1$. Clearly, $\lim_{n\to\infty} a_n = \infty$.

- $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \ldots$

  These are quotients of Fibonacci numbers $\{a_n\}$ with $a_n = \frac{F_{n+1}}{F_n}$.

  Numerically, $1, 2, 1.5, 1.667, 1.6, 1.625, 1.615, 1.619, \ldots$ Looks like $\lim_{n\to\infty} a_n$ exists and is about 1.618.

Definition 119. $\lim_{n\to\infty} a_n = L$ means that:

for every $\varepsilon > 0$ there is a value N such that, for all $n > N$, $|a_n - L| < \varepsilon$.

Here are a few basic facts about limits:

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$
- If $\lim_{x\to\infty} f(x) = L$ (the limit of a function: x is real) then $\lim_{n\to\infty} f(n) = L$ (the limit of a sequence: n is an integer).

  [The reverse is not true: for instance $\lim_{n\to\infty} \sin(\pi n) = 0$ but $\lim_{x\to\infty} \sin(\pi x)$ does not exist.]
- If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ then $\lim_{n\to\infty} (a_n + b_n) = A + B$ and $\lim_{n\to\infty} (a_n b_n) = AB$.
- If $\lim_{n\to\infty} a_n = A$ then $\lim_{n\to\infty} f(a_n) = f(A)$ provided that $f(x)$ is continuous at A.

Example 120. Since $\lim_{x\to\infty} \frac{1}{x} = 0$, we also have $\lim_{n\to\infty} \frac{1}{n} = 0$.

Example 121. Determine the following limits:

- $\lim_{n\to\infty} \frac{n-1}{n} =$
- $\lim_{n\to\infty} \frac{\ln n}{n} =$
- $\lim_{n\to\infty} n^{1/n} =$

  Note that $\ln(n^{1/n}) = \frac{1}{n}\ln(n)$ or $n^{1/n} = e^{\frac{\ln n}{n}}$. Hence, $\lim_{n\to\infty} n^{1/n} = e^L$ with $L = \lim_{n\to\infty} \frac{\ln n}{n}$.

Example 122. Just like the Fibonacci numbers, the sequence $\{A_n\}$ is defined recursively as

$$A_n = \frac{1}{n^2}[(11n^2 - 11n + 3)A_{n-1} + (n-1)^2 A_{n-2}]$$

with initial values $A_0 = 1$ and $A_1 = 3$. Compute the next few terms (called **Apéry numbers**)!

We find $A_2 = 19$, $A_3 = 147$, $A_4 = 1251$, $A_5 = 11253$. These are all integers! Why is that surprising?

[The coefficients, like 11 and 3, in the recursive definition look random. However, it seems that (essentially) there is only a finite list of possible other values that result in integers (usually, you get rational numbers). This is still an open problem. Why care? Well, all the sequences on that list have very special properties (several still unexplained) and are interestingly connected to other parts of mathematics and even physics.]

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