Example 98. Does $\int_2^3 \frac{\mathrm{d}x}{x-2}$ converge? Does $\int_2^3 \frac{\mathrm{d}x}{\sqrt{x-2}}$ converge?

Solution. Final answers: No. Yes.

Example 99. The following is VERY WRONG:

$$\text{[bad!]} \qquad \int_{-2}^{2} \frac{\mathrm{d}x}{(x+1)^2} = \left[-\frac{1}{x+1}\right]_{-2}^{2} = -\frac{1}{3} - 1 = -\frac{4}{3} \qquad \text{[bad!]}$$

[Note how even the answer is screaming trouble: we integrated something positive and got a negative value.]

What went wrong? The integrand has a problem at $x=-1$!

To be precise, it has a vertical asymptote at $x=-1$. Since $\frac{1}{(x+1)^2}$ is not differentiable (not even continuous) at $x=-1$, we can only use the antiderivative $-\frac{1}{x+1}$ for $x \neq -1$. Since -1 is in the domain of integration $[-2,2]$, we cannot directly apply the Fundamental Theorem of Calculus to this integral.

Instead, we need to split the integral into two improper integrals and analyze these individually:

$$\int_{-2}^{2} \frac{\mathrm{d}x}{(x+1)^2} = \int_{-2}^{-1} \frac{\mathrm{d}x}{(x+1)^2} + \int_{-1}^{2} \frac{\mathrm{d}x}{(x+1)^2}.$$

But $\int_{-2}^{-1} \frac{\mathrm{d}x}{(x+1)^2} = \left[-\frac{1}{x+1}\right]_{-2}^{-1}$ diverges because $\lim_{x \to -1^-} \left(-\frac{1}{x+1}\right) = \infty$. Hence, our original integral diverges.

[Note that, working on the second integral, the limit we encounter is $\lim_{x \to -1^+} \left(-\frac{1}{x+1}\right) = -\infty$. This integral, by itself, also diverges.]

Example 100. $\int \sin(x)\cos(x)\,\mathrm{d}x =$

Solution. Integrating by parts with $f(x)=\sin(x)$, $g'(x)=\cos(x)$, $g(x)=\sin(x)$, we get

$$\int \sin(x)\cos(x)\,\mathrm{d}x = \sin^2(x) - \int \cos(x)\sin(x)\,\mathrm{d}x,$$

from which we conclude that $\int \sin(x)\cos(x)\,\mathrm{d}x = \frac{1}{2}\sin^2(x) + C$.

Solution. Substitute $u=\sin(x)$, because $\mathrm{d}u = \cos(x)\,\mathrm{d}x$, to get

$$\int \sin(x)\cos(x)\,\mathrm{d}x = \int u\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2(x) + C.$$

Example 101. $\int \sin^{\lambda}(x)\cos(x)\,\mathrm{d}x =$ (with $\lambda \neq -1$)

Solution. Again, substitute $u=\sin(x)$, because $\mathrm{d}u = \cos(x)\,\mathrm{d}x$, to get

$$\int \sin^{\lambda}(x)\cos(x)\,\mathrm{d}x = \int u^{\lambda}\mathrm{d}u = \frac{1}{\lambda+1}u^{\lambda+1} + C = \frac{\sin^{\lambda+1}(x)}{\lambda+1} + C.$$

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