Partial fractions

Review. rational function $= \dfrac{\text{polynomial}}{\text{another polynomial}}$

Example 84. We are surely all familiar with putting stuff on a common demoninator like in

$$\frac{2}{x+1} + \frac{3}{x-1} = \frac{2(x-1)+3(x+1)}{(x+1)(x-1)} = \frac{5x+1}{(x+1)(x-1)}.$$

Partial fractions refers to reversing this process of putting things on a common denominator.

Example 85. The previous example allows us to compute an integral we would otherwise not be able to evaluate:

$$\int \frac{5x+1}{(x+1)(x-1)}\mathrm{d}x = \int \frac{2}{x+1}\mathrm{d}x + \int \frac{3}{x-1}\mathrm{d}x = 2\ln|x+1| + 3\ln|x-1| + C.$$

Make sure that you are very comfortable with integrating the two simpler integrals!
[For instance, notice that substituting $u = x+1$ in the first of the two, we have $\mathrm{d}u = \mathrm{d}x$ which is why we just get $\log$ of $x+1$.]

Example 86. Let us compute $\displaystyle\int \frac{1}{x(x-1)}\mathrm{d}x$ by partial fractions.

- Partial fractions tells us that $\dfrac{1}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1}$ for some numbers A, B that we still need to find.
- To find A and B we clear denominators: [i.e. multiply both sides with $x(x-1)$]

$$1 = (x-1)A + xB$$

- This equation has to be true for all values of x:
  In particular, for $x=0$: $1 = -A$.
  Likewise, for $x=1$: $1 = B$.
- We can now verify that, indeed, $\dfrac{1}{x(x-1)} = \dfrac{-1}{x} + \dfrac{1}{x-1}$.
- Finally, $\displaystyle\int \frac{1}{x(x-1)}\mathrm{d}x = \int \frac{-1}{x}\mathrm{d}x + \int \frac{1}{x-1}\mathrm{d}x = -\ln|x| + \ln|x-1| + C$.

Remark 87. An alternative (less magical) approach to finding A and B in $1 = (x-1)A + xB$ is to note that both sides are polynomials in x. Hence, they are equal if and only if all the coefficients are the same:

There is no terms involving $x^2, x^3, \ldots$ so nothing to compare there. The coefficients of x are $0 = A + B$. The constant coefficients are $1 = -A$. Combining both, we again find $A = -1$ and $B = 1$.

Example 88. Evaluate $\displaystyle\int \frac{2x-5}{x(x+1)(x+2)}\mathrm{d}x$ by partial fractions.

- Partial fractions tells us that $\dfrac{2x-5}{x(x+1)(x+2)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x+2}$ for numbers A, B, C.

  [Note that, by putting the RHS on a common denominator, we can see that the degree of the overall numerator has to be less than the degree of the denominator. It has to be checked beforehand that this is true for the LHS, too!]
- Hence: $2x - 5 = (x+1)(x+2)A + x(x+2)B + x(x+1)C$.
- Now, set $x = 0$, $x = -1$ and $x = \ldots$
- ...

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