Review 48. The (arc)length of a general curve $y = f(x)$, for $a \leqslant x \leqslant b$, is given by

$$\int_a^b \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Example 49. Our goal is to use our new technology to compute the circumference of a circle of radius r.

- Clearly, the final answer has to be $2\pi r$.
- First, in order to work with a function, we consider the (upper) half circle.
  That half-circle is described by $y = \sqrt{r^2 - x^2}$ and the arclength of that curve (from $x = -r$ to $x = r$) is half of the circumference of our circle.
- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-x}{\sqrt{r^2 - x^2}}$
- Hence, the circumference of our circle is given by the integral

$$2\int_{-r}^{r} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2}\,\mathrm{d}x = 2\int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\,\mathrm{d}x = 2\int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}}\,\mathrm{d}x.$$

- Now, we substitute $u = x/r$.
  (The motivation is that by scaling things by $1/r$ our circle gets rescaled to a circle of radius 1.)

> It is very common in applications that we need to change scales or coordinate systems. When dealing with integrals, we need to perform the corresponding substitution.
>
> This is a second very important reason why we need to be able to substitute. (So far we substituted as a means to simplify an integral.)

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{r}$, and so $\mathrm{d}x = r\,\mathrm{d}u$ [Observe that r is just a number, we can treat it like we would treat 7.]

For the boundaries of the integral: if $x = -r$, then $u = -1$. If $x = r$, then $u = 1$.

Since $x = ru$, we therefore get

$$2\int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}}\,\mathrm{d}x = 2\int_{-1}^{1} \sqrt{\frac{r^2}{r^2 - (ru)^2}}\,r\,\mathrm{d}u.$$

- Cleaning the integral up, we have $2r\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1 - u^2}}\,\mathrm{d}u$.
- We can actually evaluate this final integral (more on such integrals later) if we recall that the derivative of $\arcsin(u)$ is $1/\sqrt{1 - u^2}$. Hence,

$$2r\int_{-1}^{1} \frac{1}{\sqrt{1 - u^2}}\,\mathrm{d}u = 2r\Big[\arcsin(u)\Big]_{-1}^{1} = 2r[\arcsin(1) - \arcsin(-1)] = 2\pi r.$$

Example 50. **Work** is **force** times **distance**: $W = Fd$. Think about the work necessary to lift an object to a certain height.

Calculus comes in when the force F is not constant...

Armin Straub
straub@southalabama.edu